

PROJECTIVE GEOMETRY AND THE QUATERNIONIC FEIX–KALEDIN CONSTRUCTION

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ABSTRACT. Starting from a complex manifold S with a real-analytic c-projective structure whose curvature has type $(1, 1)$, and a complex line bundle \mathcal{L} with a connection whose curvature has type $(1, 1)$, we construct the twistor space Z of a quaternionic manifold M with a quaternionic S^1 action which contains S as a totally complex submanifold fixed by the action. This construction includes, as a special case, a construction of hypercomplex manifolds, including hyperkähler metrics on cotangent bundles, obtained independently by B. Feix [18, 19] and D. Kaledin [28, 29].

When S is a Riemann surface, M is a self-dual conformal 4-manifold, and the quotient of M by the S^1 action is an Einstein–Weyl manifold with an asymptotically hyperbolic end [26, 34], and our construction coincides with the construction presented by the first author in [9].

1. INTRODUCTION AND MAIN RESULTS

The construction of hyperkähler metrics on cotangent bundles of Kähler manifolds has a distinguished history, going back to E. Calabi’s metric on the cotangent bundle of $\mathbb{C}P^n$ [12], and its generalizations to complex semisimple Lie groups and their flag varieties [31, 32]. General constructions were provided independently by B. Feix [18] and D. Kaledin [28, 29], who showed that on a complex manifold S , any real-analytic Kähler metric induces a hyperkähler metric on a neighbourhood of the zero section in T^*S . In fact, both authors further established (see [19] in Feix’s case) that any real-analytic complex affine connection on S with curvature of type $(1, 1)$ induces a hypercomplex structure on a neighbourhood of the zero section in TS .

Herein, we develop a projective-geometric framework for these results which constructs, more generally, a quaternionic manifold with a circle action in a neighbourhood of a fixed totally complex submanifold. We begin by motivating this generalization.

1.1. Complexification and projective geometry. The “hypercomplexification” of S can be fruitfully compared (see e.g. [7]) with the complexification of a real-analytic n -manifold M : this is a holomorphic n -manifold M^c containing M as the fixed point set of an antiholomorphic involution; M^c is locally unique along M (i.e., up to unique biholomorphism between neighbourhoods of M inducing the identity on M). The underlying complex manifold $(M^c_{\mathbb{R}}, J)$ of M^c (a real $2n$ -manifold with an integrable complex structure) has M as a *totally real submanifold*, i.e., $TM \cap J(TM) = 0$, so $TM^c_{\mathbb{R}}|_M = TM \oplus J(TM)$. Since $J(TM) \cong TM$ is the normal bundle to M in $M^c_{\mathbb{R}}$, there is a local isomorphism along M between $M^c_{\mathbb{R}}$ and TM , where M is identified with the zero section in TM , along which J is an isomorphism between horizontal and vertical tangent spaces in $T(TM)$. Such a complexification of M inside TM is unique up to unique local automorphism inducing the identity to first order along M ; furthermore, R. Bielawski [7] and R. Szöke [42] show that the complexification can be determined uniquely by choosing an affine connection D on M and requiring that the tangent map

of any geodesic is holomorphic. However, the *unparametrized* geodesics of D depend only on its projective class in the following sense.

Definition 1.1. A *projective manifold* is a manifold M with *projective structure*, i.e., a *projective equivalence class* $\Pi_r = [D]_r$ of torsion-free affine connections, where $\tilde{D} \sim_r D$ if there is a 1-form $\gamma \in \Omega^1(M)$ such that for all vector fields $X, Y \in \Gamma(TM)$,

$$(1) \quad \tilde{D}_X Y = D_X Y + \llbracket X, \gamma \rrbracket^r(Y), \quad \text{where} \quad \llbracket X, \gamma \rrbracket^r(Y) = \gamma(X)Y + \gamma(Y)X.$$

The results of Bielawski and Szöke [7, 42] thus have the following consequence.

Theorem 1. A *real-analytic projective manifold* M has a *complexification* $M^c \subseteq TM$ which meets the tangent bundle to any geodesic in M in a holomorphic submanifold.

1.2. Quaternionic manifolds and totally complex submanifolds. Recall [40] that a *quaternionic structure* on a $4n$ -manifold M is a bundle \mathcal{Q} of Lie subalgebras of the endomorphism bundle $\mathfrak{gl}(TM)$ of TM which is pointwise isomorphic to the Lie algebra $\mathfrak{sp}(1)$ of imaginary quaternions acting on $\mathbb{R}^{4n} \cong \mathbb{H}^n$; a *quaternionic connection* \mathfrak{D} on (M, \mathcal{Q}) is a torsion-free affine connection preserving \mathcal{Q} . If (M, \mathcal{Q}) admits a quaternionic connection (satisfying a curvature condition when $n = 1$ which we discuss later), we say it is a *quaternionic manifold*. A submanifold S of (M, \mathcal{Q}) is *totally complex* [3] if there is a section J of $\mathcal{Q}|_S$ with $J^2 = -id$ such that:

- $J(TS) \subseteq TS$ (so that J is an almost complex structure on S);
- for all $I \in J^\perp$, $I(TS) \cap TS = 0$, where $J^\perp := \{I \in \mathcal{Q} : IJ = -JI\}$.

If M has real dimension $4n$, it follows that S has real dimension $\leq 2n$. If S is *maximal*, i.e., dimension $2n$, then $TM|_S = TS \oplus NS$ where $(NS)_u = I(T_u S)$ for any nonzero $I \in J_u^\perp$. (Any other element of J_u^\perp is a pointwise linear combination of I and IJ , so $(NS)_u$ is independent of the choice of I , and the map $J_u^\perp \times T_u S \rightarrow (NS)_u; (I, X) \mapsto IX$ induces an isomorphism $J_u^\perp \otimes_{\mathbb{C}} T_u S \cong (NS)_u$, where J_u^\perp and $T_u S$ are complex vector spaces via right multiplication by J and its left action respectively.)

Lemma 1.1. Let S be a maximal totally complex submanifold of (M, \mathcal{Q}) and \mathfrak{D} a quaternionic connection, and let $\pi : TM|_S \rightarrow TS$ be the projection along NS . Then the projection $D_X Y := \pi(\mathfrak{D}_X Y)$, for vector fields X, Y on S , defines a torsion-free complex connection (i.e., $DJ = 0$) on S , and hence J is integrable on S .

Proof. Clearly D is a torsion-free connection on S : for any vector fields X, Y on S , $D_X Y - D_Y X = \pi([X, Y]) = 0$. Furthermore,

$$\begin{aligned} (D_X J)Y &= D_X(JY) - JD_X Y = \pi(\mathfrak{D}_X(JY)) - J\pi(\mathfrak{D}_X Y) \\ &= \pi(\mathfrak{D}_X J)Y + (\pi J - J\pi)\mathfrak{D}_X Y = 0, \end{aligned}$$

since $\mathfrak{D}_X J$ is a section of J^\perp , and J commutes with π . □

If $\tilde{\mathfrak{D}}$ is another quaternionic connection on M , it is well known [2] that there is a 1-form γ on M such that $\tilde{\mathfrak{D}}_X Y = \mathfrak{D}_X Y + \llbracket X, \gamma \rrbracket^q(Y)$, where

$$(2) \quad \llbracket X, \gamma \rrbracket^q(Y) := \frac{1}{2}(\gamma(X)Y + \gamma(Y)X - \sum_{i=1}^3 (\gamma(J_i X)J_i Y + \gamma(J_i Y)J_i X))$$

where J_1, J_2, J_3 is any local anticommuting frame of \mathcal{Q} with $J_i^2 = -id$. Thus, given one quaternionic connection D , we can construct all others using $\llbracket \cdot, \cdot \rrbracket^q$.

For a maximal totally complex submanifold $S \subseteq M$, we may take the anticommuting frame defined by the given complex structure J preserving TS , a local section I of J^\perp with $I^2 = -id$, and $K = IJ$. Then for vector fields X, Y along S , we compute

$$\begin{aligned} \pi(\tilde{\mathfrak{D}}_X Y - \mathfrak{D}_X Y) &= \pi(\llbracket X, \gamma \rrbracket^q(Y)) = \llbracket X, \gamma \rrbracket^c(Y), \quad \text{where} \\ (3) \quad \llbracket X, \gamma \rrbracket^c(Y) &:= \frac{1}{2}(\gamma(Y)Z + \gamma(Z)Y - (\gamma(JY)JZ + \gamma(JZ)JY)) \end{aligned}$$

and we use $\pi(IX) = \pi(KX) = 0$. This suggests the following definition.

Definition 1.2. A *c-projective manifold* is a manifold S with an integrable complex structure J and a *c-projective structure*, i.e., an *c-projective equivalence* class $\Pi_c = [D]_c$ of torsion-free complex connections, where $\tilde{D} \sim_c D$ if there is a 1-form γ such that for all vector fields X, Y on S , $\tilde{D}_X Y = D_X Y + \llbracket X, \gamma \rrbracket^c(Y)$.

This is complex, though not necessarily holomorphic, analogue of a real projective structure (see §2.4 and [14, 24, 25, 44], some of which use misleading terms “holomorphically projective” and “h-projective”). The observations above imply the following.

Theorem 2. *Let S be a maximal totally complex submanifold of a quaternionic manifold (M, \mathcal{Q}) . Then S is a c-projective manifold, whose c-projective structure consists of the connections induced by quaternionic connections on M via Lemma 1.1.*

Since the normal bundle of S in M is isomorphic to $TS \otimes_{\mathbb{C}} J^\perp$, a neighbourhood of S in M is isomorphic to a neighbourhood of the zero section in $TS \otimes_{\mathbb{C}} J^\perp$.

We show in §2.4 that the *c-projective curvature* of S has type $(1, 1)$ with respect to J . Conversely, the quaternionic Feix–Kaledin construction exhibits every real-analytic c-projective manifold with type $(1, 1)$ c-projective curvature as a maximal totally complex submanifold of a quaternionic manifold. We next motivate this construction through the model example of quaternionic projective space.

1.3. The model example and the twistor construction. Given a quaternionic vector space $W \cong \mathbb{H}^{n+1}$, its quaternionic projectivization $M = P_{\mathbb{H}}(W) \cong \mathbb{H}P^n$ has a canonical quaternionic structure: a point $H \in M$ is a 1-dimensional quaternionic subspace of W , and its tangent space $T_H M$ is the space of quaternionic linear maps $H \rightarrow W/H$, which is itself a quaternionic vector space; the action of the imaginary quaternions on $T_H M$ defines an $\mathfrak{sp}(1)$ subalgebra $\mathcal{Q}_H \cong \mathfrak{sl}(H, \mathbb{H}) \subseteq \mathfrak{gl}(T_H M)$. Now let $W_{\mathbb{C}}$ be the underlying complex vector space of W with respect to one of its complex structures J . Then there is a natural map π_M from $Z = P(W_{\mathbb{C}}) \cong \mathbb{C}P^{2n+1}$ to M whose fibre at $H \in M$ is $P(H_{\mathbb{C}}) \cong \mathbb{C}P^1$, which is isomorphic to the 2-sphere of unit imaginary quaternions in $\mathfrak{sl}(H, \mathbb{H})$. Furthermore, these fibres are fixed by the antiholomorphic involution of Z induced by any nonzero element of J^\perp .

Now let $W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}$, where $W^{1,0} \cong W^{0,1} \cong \mathbb{C}^{n+1}$ are maximal totally complex subspaces of W with respect to the chosen complex structure J , i.e., $JW^{1,0} = W^{1,0}$, $JW^{0,1} = W^{0,1}$, and $IW^{1,0} = W^{0,1}$ for any nonzero $I \in J^\perp$. Then $P(W^{1,0})$ and $P(W^{0,1})$ are disjoint projective n -subspaces of $Z = P(W_{\mathbb{C}})$, and $S := \pi_M(P(W^{1,0})) = \pi_M(P(W^{0,1})) \cong \mathbb{C}P^n$ is a maximal totally complex submanifold of $M \cong \mathbb{H}P^n$.

Proposition 1.1. *$Z \setminus P(W^{1,0})$ is canonically isomorphic to (the total space of) the vector bundle $\text{Hom}(\mathcal{O}_{W^{0,1}}(-1), W^{1,0}) \rightarrow P(W^{0,1})$, with fibre $\text{Hom}(\tilde{x}, W^{1,0})$ over $\tilde{x} \in P(W^{0,1})$, and similarly $Z \setminus P(W^{0,1}) \cong \text{Hom}(\mathcal{O}_{W^{1,0}}(-1), W^{0,1}) \rightarrow P(W^{1,0})$. Furthermore the blow-up of Z along $P(W^{1,0}) \sqcup P(W^{0,1})$ is canonically isomorphic to the $\mathbb{C}P^1$ -bundle*

$$\hat{Z} := P(\mathcal{O}(-1, 0) \oplus \mathcal{O}(0, -1)) \rightarrow P(W^{1,0}) \times P(W^{0,1}),$$

whose fibre over (x, \tilde{x}) is $P(x \oplus \tilde{x})$.

Proof. The fibre of the map $Z \setminus P(W^{1,0}) \rightarrow P(W^{0,1}); [w + \tilde{w}] \mapsto [\tilde{w}]$ over $\tilde{x} \in P(W^{0,1})$ is $P(W^{1,0} \oplus \tilde{x}) \setminus P(W^{1,0})$. Any 1-dimensional subspace of $W^{1,0} \oplus \tilde{x}$ transverse to $W^{1,0}$ is the graph of linear map $\tilde{x} \rightarrow W^{1,0}$, yielding an isomorphism $P(W^{1,0} \oplus \tilde{x}) \setminus P(W^{1,0}) \rightarrow \text{Hom}(\tilde{x}, W^{1,0})$. The isomorphism of $Z \setminus P(W^{0,1})$ with $\text{Hom}(\mathcal{O}_{W^{1,0}}(-1), W^{0,1})$ is analogous, and the identification of the blow-up of Z with \hat{Z} follows because (see §2.2) the blow-up of a vector space E at the origin is isomorphic to the total space of the tautological bundle $\mathcal{O}_E(-1) \rightarrow P(E)$. \square

Thus Z may be obtained from $P(W^{1,0}) \times P(W^{0,1})$ by gluing together the vector bundles $\text{Hom}(\mathcal{O}_{W^{1,0}}(-1), W^{0,1}) \rightarrow P(W^{1,0})$ and $\text{Hom}(\mathcal{O}_{W^{0,1}}(-1), W^{1,0}) \rightarrow P(W^{0,1})$ to obtain a blow-down of $P(\mathcal{O}(-1, 0) \oplus \mathcal{O}(0, -1))$ along its two canonical (“zero and infinity”) sections. Each fibre $P(x \oplus \tilde{x})$ then maps to a projective line in Z with normal bundle isomorphic to $\mathcal{O}_{x \oplus \tilde{x}}(1) \otimes \mathbb{C}^{2n}$, and these are the fibres of Z over $S \subseteq M$.

This picture generalizes using an extension to quaternionic manifolds, introduced by S. Salamon [39, 40], of Penrose’s twistor theory for self-dual conformal manifolds [5, 38]. The *twistor space* of a quaternionic $4n$ -manifold (M, \mathcal{Q}) —or, for $n = 1$, a self-dual conformal manifold—is the total space Z of the 2-sphere bundle $\pi_M: Z \rightarrow M$ of elements of \mathcal{Q} which square to -1 . Salamon showed that Z admits a integrable complex structure (and hence is a holomorphic $(2n+1)$ -manifold) such that the involution ρ of Z sending J to $-J$ is antiholomorphic, and the fibres of π_M are *real twistor lines*, i.e., holomorphically embedded, ρ -invariant projective lines with normal bundle isomorphic to $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$. The following converse will be crucial to our main construction.

Theorem 3. *Let Z be a holomorphic $(2n+1)$ -manifold equipped with an antiholomorphic involution $\rho: Z \rightarrow Z$ containing a real twistor line on which ρ has no fixed points. Then the space of such real twistor lines is a $4n$ -dimensional quaternionic manifold (M, \mathcal{Q}) such that (Z, ρ) is locally isomorphic to its twistor space.*

For hyperkähler and quaternion-Kähler manifolds, this result is due to N. Hitchin et al. [23] and C. LeBrun [33] respectively. H. Pedersen and Y-S. Poon [37] establish an extension to general quaternionic manifolds, although they assume that Z is foliated by real twistor lines. However, the Kodaira deformation space [30] of u is a holomorphic $4n$ -manifold M^c with a real structure ρ_M whose fixed points are real twistor lines. It follows that the real twistor lines form a real-analytic submanifold M of M^c with real dimension $4n$, which is enough to establish the above result, following [6, 23, 33, 37].

1.4. The quaternionic Feix–Kaledin construction. Our approach (cf. [8]) to the quaternionic Feix–Kaledin construction generalizes the model example, and the approach of Feix [18, 19]. Starting from a $2n$ -manifold S equipped with an integrable complex structure J and a real-analytic c-projective structure Π_c , we build the twistor space Z of a quaternionic manifold M from a projective line bundle $\hat{Z} = P(\mathcal{L}_{1,0}^* \oplus \mathcal{L}_{0,1}^*) \xrightarrow{p} S^c$, where S^c is a complexification of S . The fibres of p over S^c are projective lines in \hat{Z} with trivial normal bundle $\mathcal{O} \otimes \mathbb{C}^{2n}$, but if we map them into a suitable blow-down Z of \hat{Z} , along “zero” and “infinity” sections $\underline{0} = P(\mathcal{L}_{1,0}^* \oplus 0)$ and $\underline{\infty} = P(0 \oplus \mathcal{L}_{0,1}^*)$, then their images in Z will have normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$.

In the model example, S^c is a product of projective spaces, and $\mathcal{L}_{1,0}$ and $\mathcal{L}_{0,1}$ are dual to tautological line bundles over the factors. In general, we show that S^c is an open subset of a projective bundle in two different ways, and the holomorphic line

bundles $\mathcal{L}_{1,0}$ and $\mathcal{L}_{0,1}$ are dual to fibrewise tautological line bundles over these projective bundles. There is some freedom in the choice of $\mathcal{L}_{1,0}$ and $\mathcal{L}_{0,1}$, which we parametrize by an auxiliary complex line bundle $\mathcal{L} \rightarrow S$ equipped with a real-analytic complex connection ∇ . In more detail, we proceed in several steps.

Step 1: Complexification. First we introduce a complexification of S , i.e., a holomorphic manifold S^c with S as the fixed point set of an antiholomorphic involution—see §2.1. Since S is a complex manifold, it has an essentially canonical complexification as the diagonal in $S^{1,0} \times S^{0,1}$, where $S^{1,0}$ denotes S with the holomorphic structure induced by J and $S^{0,1} = \overline{S^{1,0}}$ is its conjugate (with the holomorphic structure induced by $-J$) so that transposition is an antiholomorphic involution of $S^{1,0} \times S^{0,1}$. However, the c -projective structure Π_c on S and connection ∇ on \mathcal{L} may only extend to a tubular neighbourhood of the diagonal in $S^{1,0} \times S^{0,1}$, so we let S^c be such a neighbourhood, with extensions Π_c^c and ∇^c of Π_c and ∇ . Thus S^c has transverse $(0,1)$ and $(1,0)$ foliations, which are the fibres of the projections $\pi_{1,0}: S^c \rightarrow S^{1,0}$ and $\pi_{0,1}: S^c \rightarrow S^{0,1}$.

As explained in Proposition 2.4, the algebraic bracket $[\cdot, \cdot]^c$ restricts to $[\cdot, \cdot]^r$ on the leaves of the $(0,1)$ and $(1,0)$ foliations and so restrictions of connections in Π_c^c induce projective structures, and hence projective Cartan connections \mathcal{D} , along these leaves—see §2.4–§2.5. In fact, as explained in §2.6, we couple these connections to the connection ∇^c on \mathcal{L}^c to obtain connections \mathcal{D}^∇ on certain 1-jet bundles $J^1\mathcal{L}_{0,1}$ and $J^1\mathcal{L}_{1,0}$ along the leaves of the $(0,1)$ and $(1,0)$ foliations respectively.

Step 2: Development. We now introduce the fundamental assumption that Π_c and ∇ have (curvature of) type $(1,1)$ with respect to the complex structure J on S —see §2.6. By Proposition 2.5, the coupled projective Cartan connections \mathcal{D}^∇ are flat along the leaves of the $(0,1)$ and $(1,0)$ foliations. Since these leaves are assumed to be contractible, hence simply connected, the rank $n+1$ bundles $J^1\mathcal{L}_{0,1}$ and $J^1\mathcal{L}_{1,0}$ are trivialized by parallel sections along the $(0,1)$ and $(1,0)$ foliations respectively.

Definition 1.3. The bundle $\text{Aff}(\mathcal{L}_{0,1}) \rightarrow S^{1,0}$ of *affine sections* along the leaves of the $(0,1)$ foliation (the fibres of $\pi_{1,0}$) is the bundle whose fibre at $x \in S^{0,1}$ is the space of sections ℓ of $\mathcal{L}_{0,1}$ over $\pi_{1,0}^{-1}(x)$ such that $j^1\ell$ is \mathcal{D}^∇ -parallel. The bundle $\text{Aff}(\mathcal{L}_{1,0}) \rightarrow S^{0,1}$ is defined similarly. We further define $\mathcal{V}^{0,1} := \text{Aff}(\mathcal{L}_{0,1})^* \otimes \mathcal{L}_{1,0} \rightarrow S^{1,0}$ and $\mathcal{V}^{1,0} := \text{Aff}(\mathcal{L}_{1,0})^* \otimes \mathcal{L}_{0,1} \rightarrow S^{0,1}$.

The *evaluation maps* $\pi_{1,0}^* \text{Aff}(\mathcal{L}_{0,1}) \rightarrow \mathcal{L}_{0,1}$ and $\pi_{0,1}^* \text{Aff}(\mathcal{L}_{1,0}) \rightarrow \mathcal{L}_{1,0}$ over S^c send an affine section along a leaf to its value at a point on that leaf. Dual to these are line subbundles $\mathcal{L}_{0,1}^* \hookrightarrow \pi_{1,0}^* \text{Aff}(\mathcal{L}_{0,1})^*$ and $\mathcal{L}_{1,0}^* \hookrightarrow \pi_{0,1}^* \text{Aff}(\mathcal{L}_{1,0})^*$ over S^c , and hence fibrewise *developing maps* from S^c to $P(\mathcal{V}^{0,1})$ over $S^{1,0}$, or from S^c to $P(\mathcal{V}^{1,0})$ over $S^{0,1}$, sending a point of S^c to the fibre of $\mathcal{L}_{0,1}^* \otimes \mathcal{L}_{1,0}$ in $\mathcal{V}^{0,1}$, or $\mathcal{L}_{1,0}^* \otimes \mathcal{L}_{0,1}$ in $\mathcal{V}^{1,0}$ respectively. The developing maps are local diffeomorphisms, so we may assume (shrinking S^c if necessary) that they embed S^c as open subsets of $P(\mathcal{V}^{0,1})$ and $P(\mathcal{V}^{1,0})$ respectively. These induce embeddings of the line bundles $\mathcal{L}_{0,1}^* \otimes \mathcal{L}_{1,0}$ and $\mathcal{L}_{1,0}^* \otimes \mathcal{L}_{0,1}$ into the tautological line bundles $\mathcal{O}_{\mathcal{V}^{0,1}}(-1) \rightarrow P(\mathcal{V}^{0,1})$ and $\mathcal{O}_{\mathcal{V}^{1,0}}(-1) \rightarrow P(\mathcal{V}^{1,0})$.

Step 3: Blow-down. To blow \hat{Z} down along $\underline{0}$ and $\underline{\infty}$, we make following definition.

Definition 1.4. Let $\phi_{0,1}: \hat{Z} \setminus \underline{\infty} \rightarrow \mathcal{V}^{0,1}$ and $\phi_{1,0}: \hat{Z} \setminus \underline{0} \rightarrow \mathcal{V}^{1,0}$ be the restrictions, to $\hat{Z} \setminus \underline{\infty} \cong \mathcal{L}_{0,1}^* \otimes \mathcal{L}_{1,0}$ and $\hat{Z} \setminus \underline{0} \cong \mathcal{L}_{1,0}^* \otimes \mathcal{L}_{0,1}$ respectively, of the blow-downs $\mathcal{O}_{\mathcal{V}^{0,1}}(-1) \rightarrow \mathcal{V}^{0,1}$ and $\mathcal{O}_{\mathcal{V}^{1,0}}(-1) \rightarrow \mathcal{V}^{1,0}$ of zero sections of tautological line bundles.

On the complement of $\underline{0} \sqcup \underline{\infty}$, the blow-down maps $\phi_{0,1}$ and $\phi_{1,0}$ are biholomorphisms onto their image—see §2.2. However, since S^c typically embeds as a proper open subset of $P(\mathcal{V}^{0,1})$ and $P(\mathcal{V}^{1,0})$, the images of $\phi_{0,1}$ and $\phi_{1,0}$ are cones in each fibre of $\mathcal{V}^{0,1}$ and $\mathcal{V}^{1,0}$ (see Remark 2.3), hence singular along the zero sections. As a first attempt to fix this problem, we could replace these images by $\mathcal{V}^{0,1}$ and $\mathcal{V}^{1,0}$ themselves, and then glue these two vector bundles together by identifying $\phi_{0,1}(z)$ with $\phi_{1,0}(z)$ for $z \in \hat{Z} \setminus (\underline{0} \sqcup \underline{\infty})$. Unfortunately the space obtained in this way is typically not Hausdorff. We repair this by gluing instead open subsets $Z^{0,1} \subseteq \mathcal{V}^{0,1}$ and $Z^{1,0} \subseteq \mathcal{V}^{1,0}$ as follows.

Definition 1.5. Let $U^{0,1}$ and $U^{1,0}$ be tubular neighbourhoods of the zero section in $\mathcal{V}^{0,1}$ and $\mathcal{V}^{1,0}$ respectively, such that

$$(4) \quad \phi_{0,1}^{-1}(U^{0,1}) \cap \phi_{1,0}^{-1}(U^{1,0}) = \emptyset$$

and define

$$Z^{0,1} = \text{im } \phi_{0,1} \cup U^{0,1}, \quad Z^{1,0} = \text{im } \phi_{1,0} \cup U^{1,0}, \quad Z = Z^{0,1} \sqcup_{\sim} Z^{1,0},$$

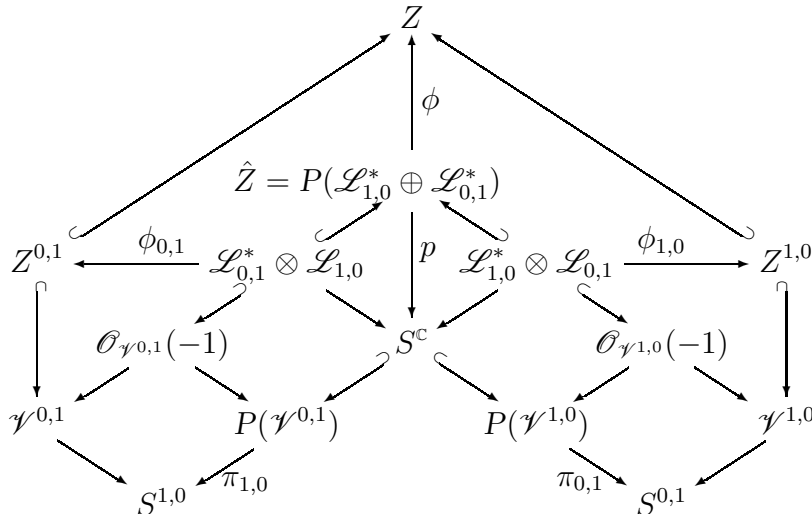
where $\phi_{0,1}(z) \sim \phi_{1,0}(z)$ for all $z \in \hat{Z} \setminus (\underline{0} \sqcup \underline{\infty})$. This gluing induces a map

$$(5) \quad \phi: \hat{Z} = P(\mathcal{L}_{1,0}^* \oplus \mathcal{L}_{0,1}^*) \rightarrow Z,$$

whose restriction to any leaf of the $(0,1)$ foliation is an isomorphism away from $\underline{0}$, and whose restriction to any leaf of the $(1,0)$ foliation is an isomorphism away from $\underline{\infty}$.

Remark 1.1. Via the developing maps, $\phi_{0,1}$ and $\phi_{1,0}$ are restrictions of the blow-down maps which contract $2n$ -dimensional zero sections of $\mathcal{L}_{0,1}^* \otimes \mathcal{L}_{1,0}$ and $\mathcal{L}_{1,0}^* \otimes \mathcal{L}_{0,1}$ to n -dimensional zero sections of $\mathcal{V}^{0,1}$ and $\mathcal{V}^{1,0}$. The multiplicative parts $(\mathcal{L}_{0,1}^* \otimes \mathcal{L}_{1,0})^\times$ and $(\mathcal{L}_{1,0}^* \otimes \mathcal{L}_{0,1})^\times$ are both isomorphic to $\hat{Z} \setminus (\underline{0} \sqcup \underline{\infty})$, the composite of these isomorphisms being the inversion map $\ell \mapsto 1/\ell$. Fibrewise, $Z^{0,1}$ and $Z^{1,0}$ look like cones with small balls added around the origin, and they are glued along the cones by inversion.

The following diagram summarizes the construction of Z , where the hooked arrows are open embeddings, and the other arrows are fibrations or blow-downs. The left-right symmetry in the diagram corresponds to interchanging the $(1,0)$ and $(0,1)$ directions.



Step 4: Canonical twistor lines. We now reach the key point of the construction. Whereas any fibre $p^{-1}(x)$ of $p: \hat{Z} = P(\mathcal{L}_{1,0}^* \oplus \mathcal{L}_{0,1}^*) \rightarrow S^c$ has trivial normal bundle in

\hat{Z} , its image $\phi(p^{-1}(x))$, called a *canonical twistor line*, has normal bundle isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$ in the blow-down Z . We thus obtain our main result.

Theorem 4. *Let (S, Π_c) be a c -projective manifold of type $(1, 1)$. Then for any complex line bundle \mathcal{L} with connection ∇ of type $(1, 1)$, the holomorphic manifold Z of Definition 1.5 is the twistor space of a quaternionic manifold M with a quaternionic S^1 action having S as a component of its fixed points. Furthermore, S is a totally complex submanifold of M , with induced c -projective structure Π_c , and a neighbourhood of S in M is S^1 -equivariantly diffeomorphic to a neighbourhood of the zero section in $TS \otimes (\mathcal{L}_{0,1}^* \otimes \mathcal{L}_{1,0})|_S$.*

Proof. • By Proposition 3.1, Z is a holomorphic manifold with a holomorphic S^1 action.
 • By Corollary 3.1, the canonical twistor lines form a family of projective lines in Z with normal bundle isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$.
 • By Proposition 3.3, ρ is an S^1 -equivariant antiholomorphic involution of Z , the canonical twistor lines parametrized by $S \subseteq S^c$ are real, and ρ has no fixed points.
 Thus Z is the twistor space of quaternionic manifold M with a quaternionic S^1 action. By Proposition 3.4, S is a (maximal) totally complex submanifold, with induced c -projective structure Π_c . The S^1 -equivariant diffeomorphism follows from Proposition 3.5, and hence S is a component of the fixed point set of the S^1 action on M . \square

Definition 1.6. The construction of Z and M in Theorem 4 from S and \mathcal{L} is called the *quaternionic Feix-Kaledin construction*.

It remains to understand when a quaternionic $4n$ -manifold (M, \mathcal{Q}) with a quaternionic S^1 action arises in this way. For this note that at any fixed point $x \in M$, the S^1 action induces a linear action on the $\mathfrak{sp}(1)$ subalgebra $\mathcal{Q}_x \subseteq \mathfrak{gl}(T_x M)$ preserving the bracket (or equivalently, the inner product). If the action is trivial, we say x is *triholomorphic*; otherwise the action is generated by a positive multiple of $[J, \cdot] \in \mathcal{Q}_x$ for some $J \in \mathcal{Q}_x$ with $J^2 = -\text{id}$ (this is a rotation fixing $\text{span}\{J\} \subseteq \mathcal{Q}_x$).

Theorem 5. *Let (M, \mathcal{Q}) be a quaternionic $4n$ -manifold with a quaternionic S^1 action whose fixed point set has a connected component S which is a submanifold of real dimension $2n$ with no triholomorphic points. Then S is totally complex, and a neighbourhood of S in M arises from the induced c -projective structure on S via the quaternionic Feix-Kaledin construction, for some complex line bundle \mathcal{L} on S .*

The rest of the paper is structured as follows. We provide background material in Section 2, then give the details of the proof of Theorem 4, and prove Theorem 5, in Section 3. Finally, Section 4 discusses examples, applications and further directions.

2. BACKGROUND ON PROJECTIVE GEOMETRY

2.1. Complexification and real structures. We first summarize some basic facts about complexification. For further information see, for example, [7] or [36, p.66].

Definition 2.1. A *real structure* ρ on a holomorphic manifold S^c is an antiholomorphic involution, i.e., an antiholomorphic map $\rho: S^c \rightarrow S^c$ with $\rho^2 = \text{id}$. If the fixed point set S of ρ is nonempty and S^c is connected, we say (S^c, ρ) is a *complexification* of S .

A *real holomorphic map* $(S_1^c, \rho_1) \rightarrow (S_2^c, \rho_2)$ between holomorphic manifolds with real structures is a holomorphic map $f: S_1^c \rightarrow S_2^c$ such that $f \circ \rho_1 = \rho_2 \circ f$.

Remark 2.1. The derivative of ρ at a fixed point $y \in S^\mathbb{C}$ is a real involution of $T_y S^\mathbb{C}$, whose ± 1 -eigenspaces are interchanged by the complex structure, hence have the same (real) dimension. It follows that the fixed point set S , if nonempty, is a real-analytic submanifold whose real dimension is the complex dimension of $S^\mathbb{C}$. Conversely, any real-analytic manifold S admits a complexification $S^\mathbb{C}$ using holomorphic extensions of real-analytic coordinates on S . Furthermore, a complexification of S is locally unique in the following sense: if $(S_1^\mathbb{C}, \rho_1)$ and $(S_2^\mathbb{C}, \rho_2)$ are both complexifications of S then there is real holomorphic isomorphism from a ρ_1 -invariant neighbourhood of S in $S_1^\mathbb{C}$ to a ρ_2 -invariant neighbourhood of S in $S_2^\mathbb{C}$.

If \mathcal{E} is a real-analytic vector bundle of rank k over a manifold S with complexification $(S^\mathbb{C}, \rho)$, then by shrinking $S^\mathbb{C}$ to a smaller connected neighbourhood of S , we may assume that the transition functions of \mathcal{E} have holomorphic extensions to $S^\mathbb{C}$ and hence construct a holomorphic vector bundle $\mathcal{E}^\mathbb{C}$ of complex rank k over $S^\mathbb{C}$, with an isomorphism $\rho^* \mathcal{E}^\mathbb{C} \cong \overline{\mathcal{E}^\mathbb{C}}$. As with the complexification $S^\mathbb{C}$ of S , $\mathcal{E}^\mathbb{C}$ is not unique, but any two complexifications of \mathcal{E} are locally isomorphic near S . Note that $TS^\mathbb{C}$ is a complexification of TS .

If \mathcal{E} is a complex vector bundle with real-analytic complex structure I , then, after shrinking $S^\mathbb{C}$ if necessary, we may assume I extends to $\mathcal{E}^\mathbb{C}$, thus defining a decomposition $\mathcal{E}^\mathbb{C} = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$ into the $\pm i$ eigenspaces of I . In particular, if $\dim S = 2n$ and J is a real-analytic almost complex structure on S , then (after shrinking $S^\mathbb{C}$ if necessary) the tangent bundle of $S^\mathbb{C}$ has a decomposition

$$TS^\mathbb{C} = T^{1,0} S^\mathbb{C} \oplus T^{0,1} S^\mathbb{C},$$

into $\pm i$ eigendistributions of J . These distributions are integrable if and only if J is an integrable complex structure, in which case $T^{1,0} S^\mathbb{C}$ and $T^{0,1} S^\mathbb{C}$ define two transverse foliations, interchanged by ρ , called the $(1,0)$ and $(0,1)$ foliations. Shrinking $S^\mathbb{C}$ if necessary, we may assume these foliations are regular, and hence define fibrations

$$\begin{array}{ccc} & S^\mathbb{C} & \\ \pi_{1,0} \swarrow & & \searrow \pi_{0,1} \\ S^{1,0} & & S^{0,1} \end{array}$$

from $S^\mathbb{C}$ to the leaf spaces $S^{1,0}$ and $S^{0,1}$ of the $(0,1)$ and $(1,0)$ foliations respectively; the real structure ρ then induces a biholomorphism $\theta: \overline{S^{0,1}} \rightarrow S^{1,0}$. We may further assume that the projections $\pi_{1,0}$ and $\pi_{0,1}$ are jointly injective, defining an embedding

$$(\pi_{1,0}, \pi_{0,1}): S^\mathbb{C} \hookrightarrow S^{1,0} \times S^{0,1}.$$

Thus we may identify $S^\mathbb{C}$ with an open subset of $S^{1,0} \times S^{0,1}$, where ρ is induced by $(x, \tilde{x}) \mapsto (\theta(\tilde{x}), \theta^{-1}(x))$, so that S is identified with the “antidiagonal” $\{(x, \theta^{-1}(x)) : x \in S^{1,0}\}$, and $T^{1,0} S^\mathbb{C} \cong TS^{1,0}$, $T^{0,1} S^\mathbb{C} \cong TS^{0,1}$ are tangent to the factors.

If $\mathcal{E} \rightarrow S$ is a complex vector bundle with an integrable $\bar{\partial}$ -operator, then (up to shrinking $S^\mathbb{C}$) the latter defines a trivialization of $\mathcal{E}^{1,0}$ along the leaves of $(0,1)$ foliation, and of $\mathcal{E}^{0,1}$ along the leaves of $(1,0)$ foliation. Thus we may write $\mathcal{E}^{1,0}$ and $\mathcal{E}^{0,1}$ as pullbacks by $\pi_{1,0}$ and $\pi_{0,1}$ of holomorphic vector bundles on $S^{1,0}$ and $S^{0,1}$ respectively.

In summary, a $2n$ -manifold S with an integrable complex structure J has an essentially canonical complexification: we may *define* $S^{1,0}$ to be S equipped with the holomorphic structure induced by J , and $S^{0,1} = \overline{S^{1,0}}$ (which has the holomorphic structure induced by $-J$) so that the biholomorphism $\theta: \overline{S^{0,1}} \rightarrow S^{1,0}$ is the identity.

Proposition 2.1. *If S has an integrable complex structure, then $S^{1,0} \times S^{0,1}$, is a complexification of S , with $\rho(x, \tilde{x}) = (\tilde{x}, x)$, and any sufficiently small complexification S^c of S may be identified with a neighbourhood of the (anti)diagonal in $S^{1,0} \times S^{0,1}$.*

A complex vector bundle $\mathcal{E} \rightarrow S$ with an integrable $\bar{\partial}$ -operator defines holomorphic vector bundles $\mathcal{E}^{1,0} \rightarrow S^{1,0}$ and $\mathcal{E}^{0,1} \rightarrow S^{0,1}$, where $\mathcal{E}^{0,1} = \overline{\theta^ \mathcal{E}^{1,0}}$, and (omitting pull-backs by $\pi_{1,0}$ and $\pi_{0,1}$) $\mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1} \rightarrow S^c$ is a complexification of $\mathcal{E} \rightarrow S$.*

Suppose that D is a real-analytic affine connection on S . Since the connection forms of D are given by real-analytic functions, we can holomorphically extend them near S to obtain a holomorphic affine connection D^c (i.e., it has holomorphic connection forms in holomorphic coordinates) on some complexification $S^c \subseteq S^{1,0} \times S^{0,1}$.

Similarly if $\mathcal{E} \rightarrow S$ admits a real-analytic complex connection ∇ compatible with the holomorphic structure, i.e., a complex connection such that $\nabla^{0,1} = \bar{\partial}_{\mathcal{E}}$, then locally we can complexify the connection (by holomorphic extension of the connection forms) to obtain a complexified connection ∇^c on \mathcal{E}^c .

2.2. Projective bundles and blow-ups. The *projective space* $P(E)$ of a vector space E is the set of 1-dimensional subspaces of E . Writing $E^\times := E \setminus \{0\}$, the map $E^\times \rightarrow P(E)$, which sends a nonzero vector to its span, realizes E^\times as the subbundle $\mathcal{O}_E(-1)^\times$ of nonzero vectors in the *tautological line bundle* $\mathcal{O}_E(-1) \rightarrow P(E)$ whose fibre at $\ell \in P(E)$ is $\mathcal{O}_E(-1)_\ell = \ell \subseteq E$.

Notation 2.1. For $k \in \mathbb{Z}$, denote $\mathcal{O}_E(k) := \mathcal{O}_E(1)^{\otimes k}$, where for any line bundle \mathcal{L} , $\mathcal{L}^{\otimes k}$ is the k -fold tensor power of \mathcal{L} for $k > 0$, with $\mathcal{L}^{\otimes 0} = \mathcal{O}$ (the trivial line bundle) and $\mathcal{L}^{\otimes k} = (\mathcal{L}^*)^{\otimes (-k)}$ for $k < 0$. We sometimes write \mathcal{L}^k for $\mathcal{L}^{\otimes k}$.

The bundle $\mathcal{O}_E(-1)$ is a subbundle of $P(E) \times E$ and the inclusion defines a tautological section of the bundle $\text{Hom}(\mathcal{O}_E(-1), E) \rightarrow P(E)$ with fibre $\text{Hom}(\mathcal{O}_E(-1), E)_\ell = \text{Hom}(\ell, E)$. Dually there is a canonical bundle map $P(E) \times E^* \rightarrow \mathcal{O}_E(1)$ (sending (ℓ, α) to $\alpha|_\ell \in \ell^*$) and hence a map from E^* to the space of global sections of $\mathcal{O}_E(1)$. The image of this map is called the space $\text{Aff}(\mathcal{O}_E(1))$ of *affine sections* of $\mathcal{O}_E(1)$ because of the following standard fact.

Observation 2.1. *The bundle map $P(E) \times E^* \rightarrow J^1 \mathcal{O}_E(1)$ induced by taking 1-jets of affine sections is a bundle isomorphism. Hence $J^1 \mathcal{O}_E(1)$ has a canonical flat (indeed, trivial) connection whose parallel sections are 1-jets of affine sections of $\mathcal{O}_E(1)$, and there is an exact sequence of bundles:*

$$(6) \quad 0 \rightarrow T^*P(E) \otimes \mathcal{O}_E(1) \rightarrow P(E) \times E^* \rightarrow \mathcal{O}_E(1) \rightarrow 0.$$

Remark 2.2. For any 1-dimensional vector space L , $P(E \otimes L)$ is canonically isomorphic to $P(E)$, but $\mathcal{O}_{E \otimes L}(-1) = \mathcal{O}_E(-1) \otimes L$. However, if $\dim E = m+1$, then by taking the top exterior power of (6), we obtain that $\mathcal{O}_E(m+1) \cong \wedge^m T^*P(E) \otimes \wedge^{m+1} E^*$.

The above ideas may be applied fibrewise to a vector bundle.

Definition 2.2. Given a vector bundle $\mathcal{E} \xrightarrow{\pi} M$, we define the *projectivization* $P(\mathcal{E}) \rightarrow M$ by requiring that for any $x \in M$, $P(\mathcal{E})_x = P(\mathcal{E}_x)$; we further define $\mathcal{E}^\times := \mathcal{E} \setminus \underline{0}$, where $\underline{0}$ is (the image of) the zero section of \mathcal{E} . This is an open subset of the *fibrewise tautological bundle* $\mathcal{O}_{\mathcal{E}}(-1) \rightarrow P(\mathcal{E})$ whose fibre over $\ell \in P(\mathcal{E})_x$ (for $x \in M$) is $\ell \leq \mathcal{E}_x$.

If $\mathcal{L} \rightarrow M$ is a line bundle, then by Remark 2.2, $P(\mathcal{E} \otimes \mathcal{L})$ is canonically isomorphic to $P(\mathcal{E})$, but $\mathcal{O}_{\mathcal{E} \otimes \mathcal{L}}(-1) \cong \mathcal{O}_{\mathcal{E}}(-1) \otimes \pi^* \mathcal{L}$.

We next summarize blow-up and blow-down, in the holomorphic category.

Definition 2.3. A map $p: \hat{M} \rightarrow M$ is called a *blow-up* of a holomorphic manifold M along a submanifold B with *exceptional divisor* $\hat{B} \subseteq \hat{M}$ if

- $p|_{\hat{B}}: \hat{B} \rightarrow B$ is isomorphic to $P(NB) \rightarrow B$, where $NB = TM|_B/TB$,
- $p|_{\hat{M} \setminus \hat{B}}: \hat{M} \setminus \hat{B} \rightarrow M \setminus B$ is a biholomorphism.

$$\begin{array}{ccc} \hat{B} & \subseteq & \hat{M} \\ \downarrow & & \downarrow p \\ B & \subseteq & M \end{array}$$

We also say M is the *blow-down* of \hat{M} along p .

The prototypical example is the blow-up of a vector space E at the origin, given by the projection $\mathcal{O}_E(-1) \hookrightarrow P(E) \times E \rightarrow E$, where the exceptional divisor is the zero section of $\mathcal{O}_E(-1) \rightarrow P(E)$. Similarly, for any vector bundle \mathcal{E} , the projection from $\mathcal{O}_{\mathcal{E}}(-1)$ to \mathcal{E} blows down the zero section of $\mathcal{O}_{\mathcal{E}}(-1)$ to the zero section of \mathcal{E} .

These examples have a further variant to *projective completions* such as the projective line bundle $P(\mathcal{O} \oplus \mathcal{O}_E(-1)) \rightarrow P(E)$. This is a subbundle of $P(E) \times P(\mathbb{C} \oplus E)$, with fibre $P(\mathbb{C} \oplus \ell) \subseteq P(\mathbb{C} \oplus E)$ over $\ell \in P(E)$. Hence there is a blow-down map $P(\mathcal{O} \oplus \mathcal{O}_E(-1)) \rightarrow P(\mathbb{C} \oplus E)$ which is isomorphic to the blow-down $\mathcal{O}_E(-1) \rightarrow E$ on the complement of the section $P(\mathcal{O}_E(-1)) \cong P(E)$. We shall later use the following.

Observation 2.2. In the blow-down $P(\mathcal{O} \oplus \mathcal{O}_E(-1)) \rightarrow P(\mathbb{C} \oplus E)$, the fibre $P(\mathbb{C} \oplus \ell)$ over $\ell \in P(E)$ maps to the corresponding projective line in $P(\mathbb{C} \oplus E)$, with normal bundle $TP(\mathbb{C} \oplus E)|_{P(\mathbb{C} \oplus \ell)}/TP(\mathbb{C} \oplus \ell) \cong \mathcal{O}_{\mathbb{C} \oplus \ell}(1) \otimes E/\ell$.

Here the normal bundle is identified by applying (6) to $P(\mathbb{C} \oplus \ell)$ and $P(\mathbb{C} \oplus E)$.

Remark 2.3. As this last example illustrates, blow-up and blow-down are local to the submanifold or exceptional divisor. Hence disconnected submanifolds and exceptional divisors can be blown up or down componentwise. On the other hand, the blow-down of the inverse image of an open subset $U \subseteq P(E)$ in $\mathcal{O}_E(-1)$ (for example) is the cone on U in E , which (for U proper) is singular at the origin.

2.3. Cartan geometries. Let G be a real or complex Lie group, and P a (closed) Lie subgroup, so that G/P is a (smooth or holomorphic) homogeneous space. Let M be a (smooth or holomorphic) manifold with the same dimension as G/P .

Definition 2.4. A *Cartan connection of type (G, P)* on M is a principal G -bundle $\mathcal{G} \rightarrow M$, with a principal G -connection $\eta: T\mathcal{G} \rightarrow \mathfrak{g}$ and a reduction $\iota: \mathcal{P} \hookrightarrow \mathcal{G}$ of structure group to $P \leq G$ satisfying the following (open) *Cartan condition*:

- the pullback $\iota^*\eta$ induces a bundle isomorphism of $T\mathcal{P}$ with $\mathcal{P} \times \mathfrak{g}$.

A manifold M with a Cartan connection is called a *Cartan geometry*. Its *Cartan bundle* is the bundle of homogeneous spaces $\mathcal{C}_M := \mathcal{G}/P \cong \mathcal{G} \times_G (G/P) \cong \mathcal{G} \times_P (G/P)$ over M . The principal connection η on \mathcal{G} induces a connection on \mathcal{C}_M , while the reduction to P equips \mathcal{C}_M with a *tautological section* $\tau: M \cong \mathcal{P}/P \hookrightarrow \mathcal{G}/P = \mathcal{C}_M$.

The model Cartan connection of type (G, P) is the reduction $G \hookrightarrow (G/P) \times G$ of principal bundles over G/P , with connection given by the Maurer–Cartan form $\eta_G: TG \rightarrow \mathfrak{g}$ of G . This is an isomorphism on each tangent space, so the bundle map $T(G/P) \rightarrow G \times_P (\mathfrak{g}/\mathfrak{p})$, induced by the horizontal, P -equivariant 1-form $\eta_G + \mathfrak{p}: TG \rightarrow \mathfrak{g}/\mathfrak{p}$, is a bundle isomorphism.

For a general Cartan geometry M of type (G, P) , it follows that the vertical bundle of \mathcal{C}_M is (isomorphic to) $\mathcal{G} \times_G T(G/P) \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$, and the induced connection on \mathcal{C}_M is the 1-form $\eta_{\mathcal{C}}: T\mathcal{C}_M \rightarrow \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ induced by the (horizontal, P -equivariant) 1-form $\eta + \mathfrak{p}: T\mathcal{G} \rightarrow \mathfrak{g}/\mathfrak{p}$. Let $\mathfrak{g}_M = \mathcal{G} \times_G \mathfrak{g} \cong \mathcal{P} \times_P \mathfrak{g}$ and $\mathfrak{p}_M = \mathcal{P} \times_P \mathfrak{p}$. Then the covariant derivative $\eta_M := \tau^* \eta_{\mathcal{C}}: TM \rightarrow \mathcal{P} \times_P (\mathfrak{g}/\mathfrak{p}) \cong \mathfrak{g}_M/\mathfrak{p}_M$ of the tautological section τ is the 1-form on M induced by the pullback $\iota^*(\eta + \mathfrak{p}) = \iota^* \eta + \mathfrak{p}: T\mathcal{P} \rightarrow \mathfrak{g}/\mathfrak{p}$. The Cartan condition means (equivalently) that η_M is a bundle isomorphism.

The key idea behind Cartan connections is that if \mathcal{D} is flat, then in a local trivialization \mathcal{C}_M by parallel sections over an open subset U , the tautological section $\tau|_U: U \rightarrow \mathcal{C}|_U \cong U \times G/P$ defines a *developing map* from U to G/P : by the Cartan condition, these maps are local diffeomorphisms, which identify the universal cover of M with a cover of an open subset of G/P . Since this notion of development will be crucial to us, we establish it explicitly in the case we need, using a linear representation of the Cartan connection, described in §2.5.

2.4. Projective parabolic geometries. Smooth projective, c-projective and quaternionic manifolds are Cartan geometries modelled on the projective spaces $\mathbb{R}P^n$, $\mathbb{C}P^n$ and $\mathbb{H}P^n$, which are (real) homogeneous spaces for the projective general linear groups $PGL(n, \mathbb{R})$, $PGL(n, \mathbb{C})$ and $PGL(n, \mathbb{H})$. The corresponding holomorphic Cartan geometries are modelled on complexifications of these varieties, namely $\mathbb{C}P^n$, $\mathbb{C}P^n \times \mathbb{C}P^n$ and the grassmannian $Gr_2(\mathbb{C}^{2(n+1)})$ of two dimensional subspaces of $\mathbb{C}^{2(n+1)}$.

These Cartan geometries are examples of *parabolic geometries* [16]: the model G/P is a *generalized flag variety*, with G semisimple, and \mathfrak{p} a parabolic subalgebra of \mathfrak{g} . This means that the Killing perp \mathfrak{p}^\perp is a nilpotent ideal on \mathfrak{p} —indeed, in the above examples, \mathfrak{p}^\perp is abelian. For such Cartan geometries, the isomorphism $TM \cong \mathfrak{g}_M/\mathfrak{p}_M$ induces an isomorphism of $T^*M \cong \mathfrak{p}_M^\perp := \mathcal{P} \times_P \mathfrak{p}^\perp$, the Lie bracket on \mathfrak{g}_M induces a graded Lie bracket $[\cdot, \cdot]$ on $TM \oplus (\mathfrak{p}_M/\mathfrak{p}_M^\perp) \oplus T^*M$, and so there is an algebraic bracket

$$[\cdot, \cdot]: TM \times T^*M \rightarrow \mathfrak{p}_M/\mathfrak{p}_M^\perp \subseteq \mathfrak{gl}(TM).$$

These geometries all admit an equivalent class Π of torsion-free connections $[D]$, where

$$\tilde{D} \sim D \quad \Leftrightarrow \quad \exists \gamma \in \Omega^1(M) \quad \text{such that} \quad \tilde{D}_X Y = D_X Y + [[X, \gamma]](Y)$$

for all vector fields X, Y . For projective, quaternionic and c-projective manifolds, the bracket is defined explicitly in equations (1), (2) and (3) respectively.

If one regards the curvature R^D as a function of $D \in \Pi$, then its derivative with respect to a 1-form γ is $\partial_\gamma R^D = -[[id \wedge D\gamma]]$, where $\partial_\gamma F(D) = \frac{d}{dt} F(D + t\gamma)|_{t=0}$ and $[[id \wedge D\gamma]]_{X,Y} = [[X, D_Y \gamma]] - [[Y, D_X \gamma]]$. One further feature of these geometries is the existence of a “normalized Ricci” or “Rho” tensor $r^D \in \Omega^1(TM)$ such that $\partial_\gamma r^D = -D\gamma$ and hence $W := R^D - [[id \wedge r^D]]$ is an invariant of the geometry (i.e., independent of $D \in \Pi$) called its *Weyl curvature*. It follows also that the *Cotton–York curvature* $C^D := d^D r^D$ satisfies $\partial_\gamma C^D = -d^D D\gamma + [[[id, \gamma]] \wedge r^D] = -[[W, \gamma]]$. In particular, if the Weyl curvature vanishes, then the Cotton–York curvature is an invariant.

Conversely, given an equivalence class Π of torsion-free affine connections on M , compatible with an appropriate reduction of the frame bundle, one can construct a Cartan connection η which is flat if and only if the Weyl and Cotton–York curvatures vanish. This is part of the general theory of parabolic geometries [16], but we only need the construction for projective structures, which we now discuss.

2.5. Projective structures, affine sections and development. On a projective space $P(E)$, the trivialization $J^1\mathcal{O}_E(1) \cong P(E) \times E^*$ of Observation 2.1 may be viewed as a linear representation of a flat Cartan connection. Its parallel sections are 1-jets of sections of $\mathcal{O}_E(1)$ induced by linear forms on E , which are affine functions in any affine chart. Globally, these are the elements of the space $H^0(P(E), \mathcal{O}_E(1))$ of regular (or holomorphic) sections. Locally, these *affine sections* of $\mathcal{O}_E(1)$ are solutions of a second order differential equation. Projective structures generalize this local description.

Definition 2.5. Let M be a smooth or holomorphic n -manifold. Then we denote by $\mathcal{O}_M(1)$ a (chosen) line bundle over M that satisfies $\mathcal{O}_M(n+1) := \mathcal{O}_M(1)^{\otimes(n+1)} \cong \wedge^n TM$. We set $\mathcal{O}_M(-1) := (\mathcal{O}_M(1))^*$.

Let Π_r be a projective structure on a manifold M . A choice of $D \in \Pi_r$ gives a splitting of the 1-jet sequence

$$0 \rightarrow T^*M \otimes \mathcal{O}_M(1) \rightarrow J^1\mathcal{O}_M(1) \rightarrow \mathcal{O}_M(1) \rightarrow 0,$$

i.e., an isomorphism $J^1\mathcal{O}_M(1) \cong \mathcal{O}_M(1) \oplus (T^*M \otimes \mathcal{O}_M(1))$ sending $j^1\ell$ to $(\ell, D\ell)$. For $n > 1$, there is also a normalized Ricci tensor r^D associated to D , with $\partial_\gamma r^D = -D\gamma$.

Definition 2.6. For any $D \in \Pi_r$, $\ell \in \mathcal{O}_M(1)$ and $\alpha \in T^*M \otimes \mathcal{O}_M(1)$, let $\begin{bmatrix} \ell \\ \alpha \end{bmatrix}_D = j^1\ell - D\ell + \alpha$ (defined using a local extension of ℓ) be the element of $J^1\mathcal{O}_M(1)$ corresponding to $(\ell, \alpha) \in (\mathcal{O}_M(1) \oplus T^*M \otimes \mathcal{O}_M(1))$. Define a connection \mathcal{D} on $J^1\mathcal{O}_M(1)$ by

$$(7) \quad \mathcal{D}_X \begin{bmatrix} \ell \\ \alpha \end{bmatrix}_D = \begin{bmatrix} D_X\ell - \alpha(X) \\ D_X\alpha + (r^D)_X\ell \end{bmatrix}_D.$$

Proposition 2.2. *The connection \mathcal{D} does not depend on the choice of $D \in \Pi_r$.*

Proof. Since $\partial_\gamma D\ell = \gamma\ell$, we have

$$\partial_\gamma \begin{bmatrix} \ell \\ \alpha \end{bmatrix}_D = \partial_\gamma(j^1\ell - D\ell + \alpha) = -\gamma\ell = \begin{bmatrix} 0 \\ -\gamma\ell \end{bmatrix}_D.$$

Then by the Leibniz rule

$$\partial_\gamma \begin{bmatrix} D_X\ell - \alpha(X) \\ D_X\alpha + (r^D)_X\ell \end{bmatrix}_D = - \begin{bmatrix} 0 \\ \gamma(D_X\ell - \alpha(X)) \end{bmatrix}_D + \begin{bmatrix} \gamma(X)\ell \\ \llbracket X, \gamma \rrbracket^r \cdot \alpha - D_X\gamma\ell \end{bmatrix}_D.$$

Since α is $\mathcal{O}_M(1)$ -valued 1-form, $\llbracket X, \gamma \rrbracket^r \cdot \alpha = -\alpha(X)\gamma$, and hence

$$\partial_\gamma \begin{bmatrix} D_X\ell - \alpha(X) \\ D_X\alpha + (r^D)_X\ell \end{bmatrix}_D = \begin{bmatrix} \gamma(X)\ell \\ -\gamma D_X\ell - (D_X\gamma)\ell \end{bmatrix}_D = \mathcal{D}_X \begin{bmatrix} 0 \\ -\gamma\ell \end{bmatrix}_D.$$

Thus $\partial_\gamma \circ \mathcal{D} = \mathcal{D} \circ \partial_\gamma$ on $J^1\mathcal{O}_M(1)$, which completes the proof. \square

Definition 2.7. A section ℓ of $\mathcal{O}_M(1)$ over M is called an *affine section* if $j^1\ell$ is a \mathcal{D} -parallel section of $J^1\mathcal{O}_M(1)$. Note that if $\begin{bmatrix} \ell \\ \alpha \end{bmatrix}_D$ is parallel for \mathcal{D} then $\alpha = D\ell$, i.e., $\begin{bmatrix} \ell \\ \alpha \end{bmatrix}_D = j^1\ell$, and hence $D^2\ell + r^D\ell = 0$. Thus $\ell \mapsto j^1\ell$ is a bijection between affine sections of $\mathcal{O}_M(1)$ and \mathcal{D} -parallel sections of $J^1\mathcal{O}_M(1)$.

Proposition 2.3. *\mathcal{D} is flat iff Π_r has vanishing Weyl and Cotton–York curvatures.*

Proof. Choosing $D \in \Pi_r$ and computing the curvature of \mathcal{D} from (7), we obtain

$$R_{X,Y}^{\mathcal{D}} \begin{bmatrix} \ell \\ \alpha \end{bmatrix}_D = \begin{bmatrix} 0 \\ W_{X,Y} \cdot \alpha + C_{X,Y}^D \ell \end{bmatrix}_D$$

for all vector fields X, Y , where we use the fact that $\text{tr}(W_{X,Y}) = 0$. \square

If $n > 2$ and $W = 0$, the differential Bianchi identity implies that $C^D = d^D r^D = 0$, and so \mathcal{D} is flat if and only if the projective Weyl curvature vanishes. For $n = 2$, W is identically zero, and so \mathcal{D} is flat if and only if the projective Cotton–York curvature (which is a projective invariant, also known as the Liouville tensor) vanishes.

Remark 2.4. If \mathcal{L} is a line bundle with connection ∇ on a projective manifold M , then we can define a coupled (tensor product) connection \mathcal{D}^∇ on $J^1\mathcal{O}_M(1) \otimes \mathcal{L}$, and the map $\ell \otimes u \mapsto (j^1\ell) \otimes u + \ell \otimes \nabla u$ similarly defines a bijection between distinguished “affine sections” of $\mathcal{O}_M(1) \otimes \mathcal{L}$ and \mathcal{D}^∇ -parallel sections of $J^1\mathcal{O}_M(1) \otimes \mathcal{L}$.

2.6. C-projective structures and their foliations. Let (S, J) be a complex manifold of complex dimension $n > 1$, and let Π_c be a real-analytic c-projective structure on S (i.e., there is a real-analytic connection in Π_c). Then we can extend real-analytic connections in Π_c to a complexification S^c of (S, J) as in §2.1. Since $[\cdot, \cdot]$ depends only on J , it extends to any such complexification, the following is immediate.

Observation 2.3. *There is a complexification (S^c, Π_c^c) of (S, J, Π_c) such that the holomorphic connections in Π_c^c are holomorphic extensions of connections in Π_c . The c-projective Weyl and Cotton–York curvatures of Π_c^c are holomorphic extensions of corresponding c-projective Weyl and Cotton–York curvatures of Π_c .*

Proposition 2.4. *A holomorphic c-projective structure Π_c^c on $S^c \hookrightarrow S^{1,0} \times S^{0,1}$ induces holomorphic projective structures on the leaves of the $(1, 0)$ and $(0, 1)$ foliations.*

Proof. Since $TS^c = TS^{1,0} \oplus TS^{0,1}$, any connection in Π_c^c induces a connection on any leaf by restriction and projection. Now vectors tangent to the $(1, 0)$ and $(0, 1)$ foliations are of the form $X + iJX$ and $X - iJX$ respectively, and for any 1-form γ on S^c ,

$$\begin{aligned} [X + iJX, \gamma]^c(Y + iJY) &= [X + iJX, \gamma]^r(Y + iJY), \\ [X - iJX, \gamma]^c(Y - iJY) &= [X - iJX, \gamma]^r(Y - iJY). \end{aligned}$$

Hence c-projectively related connections on S^c , after restriction to leaves of the $(1, 0)$ and $(0, 1)$ foliations, are projectively related. \square

Remark 2.5. Conversely the projective structures on the leaves determine Π_c : for any $y \in S^c$ and any affine connections D and \tilde{D} on the leaves through y , there is a unique affine connection at y preserving the product structure and restricting to D and \tilde{D} .

Since the decomposition $TS^c = TS^{1,0} \oplus TS^{0,1}$ is a holomorphic extension of the type decomposition $TS \otimes \mathbb{C} = T^{1,0}S \oplus T^{0,1}S$ on S , the decomposition

$$\wedge^2 T^*S^c = \wedge^2 T^*S^{1,0} \oplus (T^*S^{1,0} \otimes T^*S^{0,1}) \oplus \wedge^2 T^*S^{0,1}$$

is a holomorphic extension of the type decomposition $\wedge^2 T^*S \otimes \mathbb{C} = \wedge^{2,0} \oplus \wedge^{1,1} \oplus \wedge^{0,2}$.

Definition 2.8. We say that a c-projective structure Π_c on (S, J) has *type* $(1, 1)$ if its c-projective Weyl and Cotton–York curvatures have type $(1, 1)$.

Proposition 2.5. *A real-analytic c-projective structure of type $(1, 1)$ induces flat projective structures on the leaves of $(1, 0)$ and $(0, 1)$ foliations in any complexification.*

Proof. As the c-projective Weyl and Cotton–York have type $(1, 1)$, their holomorphic extensions have vanishing pullbacks, as 2-forms, to any leaf of the $(1, 0)$ or $(0, 1)$ foliation. However, due to the relation between the algebraic brackets in the proof of Proposition 2.4, these pullbacks are the projective Weyl and Cotton–York curvatures of the leaves, so the induced projective structures are flat by Proposition 2.3. \square

We now explain the role of the line bundle $\mathcal{L} \rightarrow S$ with connection ∇ , whose holomorphic extension ∇^c to S^c provides line bundles with connection along the $(1, 0)$ and $(0, 1)$ foliations which we can use to twist the projective Cartan connections along the leaves as in Remark 2.4. To preserve flatness of the leafwise projective structures, we require that ∇^c is flat along leaves, i.e., ∇ has type $(1, 1)$ curvature. In particular, $\nabla^{0,1}$ is a holomorphic structure on \mathcal{L} .

For a simply-connected projective manifold, a twist by a flat line bundle is essentially trivial, corresponding to the ambiguity in $\mathcal{O}_E(1) \rightarrow P(E) = P(E \otimes L)$ mentioned in Remark 2.2. However, here we have two families of projective leaves, and ambiguities in the choice of $\mathcal{O}(1)$ along these leaves which need not be compatible—and which we want to encode in the $(1, 1)$ curvature of ∇ . In order to do this, we introduce holomorphic line bundles $\mathcal{L}_{1,0}$ and $\mathcal{L}_{0,1}$ over S^c which are pullbacks, by $\pi_{1,0}$ and $\pi_{0,1}$ respectively, of holomorphic line bundles, also denoted $\mathcal{L}_{1,0}$ and $\mathcal{L}_{0,1}$, on $S^{1,0} = (S, J)$ and $S^{0,1} = \overline{S^{1,0}} = (S, -J)$ respectively. We require that $\mathcal{L}_{1,0}$ and $\mathcal{L}_{0,1}$ are conjugate, and that $(\mathcal{L} \otimes \mathcal{L}_{1,0}^*)^{\otimes(n+1)}$ is isomorphic to the canonical bundle $\wedge^n T^* S^{1,0}$ of $S^{1,0}$.

2.7. Complexified quaternionic structures and α -submanifolds. Let Z be a twistor space for a quaternionic manifold [40, 23, 33, 37], i.e., a holomorphic $(2n+1)$ -manifold with a real structure (antiholomorphic involution) $\rho: Z \rightarrow Z$, admitting a *twistor line* (a holomorphically embedded projective line with normal bundle isomorphic to $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$) which is real, i.e., ρ -invariant, and on which ρ has no fixed points.

By Kodaira deformation theory [30], the moduli space of twistor lines in Z is a holomorphic $4n$ -manifold M^c , and there is an incidence relation or correspondence

$$(8) \quad \begin{array}{ccc} & F_M := \{(z, u) \in Z \times M^c : z \in u\} & \\ \swarrow \pi_Z & & \searrow \pi_{M^c} \\ Z & & M^c, \end{array}$$

where we identify $u \in M^c$ with the corresponding twistor line $u \subseteq Z$. The fibre of the incidence space F_M over $u \in M^c$ is essentially u ; thus F_M “separates twistor lines” (the fibres are disjoint). The normal bundles to the twistor lines define a bundle $\mathcal{N} \rightarrow F_M$ with fibre

$$\mathcal{N}_{(z,u)} := T_z Z / T_z u.$$

We then have, again by Kodaira [30], that

$$T_u M^c \cong H^0(u, \mathcal{N}|_u).$$

Locally over M^c , we may decompose \mathcal{N} (noncanonically) as $\mathcal{N} = \pi_{M^c}^* \mathcal{E} \otimes \pi_Z^* \mathcal{O}_Z(1)$ where \mathcal{E} is a rank $2n$ bundle on M^c and $\mathcal{O}_Z(1)$ is a line bundle on Z restricting to a dual tautological bundle on each twistor line. Hence

$$TM^c \cong \mathcal{E} \otimes \mathcal{H},$$

where $\mathcal{H}_u = H^0(u, \mathcal{O}_Z(1)|_u)$, so that $F_M \rightarrow M^c$ is canonically isomorphic to $P(\mathcal{H}^*) \cong P(\mathcal{H})$ (since \mathcal{H} has rank two), and we have used the fact that $\pi_{M^c}^* \mathcal{E}|_u = u \times \mathcal{E}_u$. This tensor decomposition of TM^c is the fundamental structure carried by M^c [37, 6], although \mathcal{E}, \mathcal{H} are only determined up to tensoring by mutually inverse line bundles. The quaternionic connections on M^c are the tensor product connections on $TM^c \cong \mathcal{E} \otimes \mathcal{H}$ which are torsion-free. We say a tangent vector to M^c is *null* if it is decomposable in $\mathcal{E} \otimes \mathcal{H}$.

The fibre of F_M over $z \in Z$ projects to a submanifold α_z of M^c called an α -submanifold. Thus $u \in \alpha_z$ iff $z \in u$, and then $T_u \alpha_z = \mathcal{E}_u \otimes \mathcal{O}_Z(-1)_z$, so that tangent spaces to α_z are null. Since the normal bundle to u has degree 1, the twistor lines through $z \in u$ are determined by their tangent space at z . Thus α_z is isomorphic to an open submanifold of $P(T_z Z)$, and has a canonical flat projective structure: any $\Theta \in Gr_{k+1}(T_z Z)$ parametrizes a k -dimensional projective (totally geodesic) submanifold of α_z given by the twistor lines tangent to Θ at z .

Any such null projective k -submanifold of M^c is determined by its tangent space at a point u , which is a subspace of the form $\theta \otimes \ell \subseteq \mathcal{E}_u \otimes \mathcal{H}_u = T_u M$ where θ is a k -dimensional subspace of \mathcal{E}_u and ℓ is a 1-dimensional subspace of \mathcal{H}_u . The tangent lifts of null projective k -submanifolds thus foliate the subbundle $Gr_k(\mathcal{E}) \times_{M^c} P(\mathcal{H})$ of null k -planes in $Gr_k(TM) \cap P(\wedge^k \mathcal{E} \otimes S^k \mathcal{H}) \hookrightarrow P(\wedge^k TM)$ over the grassmannian bundle $Gr_{k+1}(TZ)$ as follows.

$$\begin{array}{ccccc}
 & & Gr_k(\mathcal{E}) \times_{M^c} P(\mathcal{H}) & \hookrightarrow & P(\wedge^k \mathcal{E} \otimes S^k \mathcal{H}) & \hookrightarrow & P(\wedge^k TM^c) \\
 & \swarrow & \downarrow & & \downarrow & & \swarrow \\
 Gr_{k+1}(TZ) & & P(\mathcal{H}) & & & & \\
 \downarrow & \swarrow \pi_Z & \searrow \pi_{M^c} & & & & \\
 Z & & & & M^c & &
 \end{array}$$

For $k = 1$, the geodesics of these projective structures are called *null geodesics* of M^c . At the other extreme, when $k = 2n - 1$, $Gr_{2n}(TZ) \cong P(T^*Z)$ and $Gr_{2n-1}(\mathcal{E}) \cong P(\mathcal{E}^*)$.

Proposition 2.6. *On any α -submanifold α_z in a complexified quaternionic manifold M^c , any quaternionic connection \mathfrak{D} induces an affine connection on α_z compatible with its canonical flat projective structure.*

Proof. Observe that $\pi_Z^{-1}(z)$ is the image of a section of $P(\mathcal{H})|_{\alpha_z}$ and if h is a nonvanishing lift of this section to $\mathcal{H}|_{\alpha_z}$, then any vector (field) tangent to α_z have the form $X = e \otimes h$ for an element (or section) e of $\mathcal{E}|_{\alpha_z}$. Since \mathfrak{D} is torsion-free, and isomorphic to $\mathfrak{D}^{\mathcal{E}} \otimes \mathfrak{D}^{\mathcal{H}}$, we have, for any two null vector fields $X_1 = e_1 \otimes h_1$ and $X_2 = e_2 \otimes h_2$,

$$(9) \quad [X_1, X_2] = \mathfrak{D}_{X_1}^{\mathcal{E}} e_2 \otimes h_2 - \mathfrak{D}_{X_2}^{\mathcal{E}} e_1 \otimes h_1 + e_2 \otimes \mathfrak{D}_{X_1}^{\mathcal{H}} h_2 - e_1 \otimes \mathfrak{D}_{X_2}^{\mathcal{H}} h_1.$$

If $h_1 = h_2 = h$, then $[X_1, X_2]$ is tangent to α_z for all e_1, e_2 , so $\mathfrak{D}_X^{\mathcal{H}}$ preserves the span of h for all X tangent to α_z . Hence \mathfrak{D} restricts to a (torsion-free) connection on α_z .

It remains to show that \mathfrak{D} preserves any projective hypersurface of α_z , i.e., the submanifold of twistor lines tangent to any hyperplane in $T_z Z$. Such twistor lines generate a hypersurface \mathcal{Y} in Z , and the twistor lines in \mathcal{Y} form a codimension two submanifold Y of M^c , with conormal bundle $\varepsilon \otimes \mathcal{H}^*$, where ε is a line subbundle of \mathcal{E}^* over Y . Now equation (9) implies that $\mathfrak{D}_X^{\mathcal{E}}$ preserves $\ker \varepsilon$ along Y for X tangent to Y . Hence $Y \cap \alpha_z$ is totally geodesic with respect to \mathfrak{D} . \square

2.8. C-projective surfaces, projective curves and conformal geometry. A complex structure J on an oriented surface S is the same data as a conformal structure on S , and complex connections are conformal connections. Torsion-free conformal (i.e., complex) connections on (S, J) form an affine space modelled on 1-forms, forming a unique c-projective class on S . However, these data do not suffice to construct a Cartan connection modelled on the generalized flag variety $S^2 \cong \mathbb{CP}^1$ for $SO_0(3, 1) \cong PSL(2, \mathbb{C})$, so we need to modify the notion of a c-projective or conformal (Möbius) structure. Similarly a (real or holomorphic) projective curve C has a unique projective class of affine

connections, but these do not determine the second order *Hill operator* on $\mathcal{O}_C(1)$ whose kernel consists of the affine sections.

Following [13], we therefore require that (S, J) is equipped with a tracefree hessian operator (or *Möbius structure*), which is a second order differential operator $\mathcal{H}: \Gamma L \rightarrow \Gamma \mathcal{S}_0^2 T^* S$, where $L := \mathcal{O}_S(1)$ is a square root of $\wedge^2 TS$, such that for some (hence any) torsion-free connection D there is a section r_0^D of $\mathcal{S}_0^2 T^* S$ with

$$\mathcal{H}(\ell) = \text{sym}_0 D^2 \ell + r_0^D \ell$$

for all sections ℓ of L . This allows us to construct a normalized Ricci tensor r^D with $\partial_\gamma r^D = -D\gamma$, which is the crucial ingredient to build a Cartan connection.

Assuming \mathcal{H} is real-analytic, it extends to a complexification $S^c \hookrightarrow S^{1,0} \times S^{0,1}$, with

$$\begin{aligned} \mathcal{S}_0^2 T^* S^c &= (T^* S^{1,0})^2 \oplus (T^* S^{0,1})^2, \\ (r_0^D)^c &= (r_0^D)^{(2,0)} \oplus (r_0^D)^{(0,2)}. \end{aligned}$$

We now define, as in §2.5–§2.6, a connection along the leaves of the $(1, 0)$ foliation by

$$\mathcal{D}_Y^{1,0} \begin{bmatrix} \ell \\ \alpha \end{bmatrix}_D = \begin{bmatrix} D_Y^{1,0} \ell - \alpha(Y) \\ D_Y^{1,0} \alpha + (r_0^D)^{(2,0)} \ell \end{bmatrix}_D,$$

where ℓ is a section of $\mathcal{O}(1)$, α is an $\mathcal{O}(1)$ -valued $(1, 0)$ -form and Y is a $(1, 0)$ -vector field. As in Proposition 2.2, $\mathcal{D}^{1,0}$ is independent of the choice of D , and a similar construction applies along the leaves of the $(0, 1)$ foliation.

3. DETAILS AND PROPERTIES OF THE CONSTRUCTION

3.1. The twistor space. We now fill in the remaining details in the proof of Theorem 4. First, we need to show that $U^{1,0}$ and $U^{0,1}$ can be chosen so that Z , constructed in Definition 1.5 is a twistor space with a holomorphic S^1 action.

Proposition 3.1. *Z is a complex manifold, with a holomorphic vector field induced by scalar multiplication by $\lambda \in \mathbb{C}^\times$ in the fibres of $\mathcal{V}^{0,1}$ and by λ^{-1} in the fibres of $\mathcal{V}^{1,0}$.*

Proof. As Z is obtained by gluing open subsets of the manifolds $Z^{0,1} \subseteq \mathcal{V}^{0,1}$ and $Z^{1,0} \subseteq \mathcal{V}^{1,0}$ by a relation intertwining the action of λ and λ^{-1} , it remains to show that Z is Hausdorff. So suppose $z \in Z^{1,0}$ and $\tilde{z} \in Z^{0,1}$ with $[z] \neq [\tilde{z}]$ in Z . If $z \in \text{im } \phi_{1,0}$ or $\tilde{z} \in \text{im } \phi_{0,1}$ then we can replace it by the corresponding point in $Z^{0,1}$ or $Z^{1,0}$, which is distinct, hence separated, from \tilde{z} or z . However, for $z \in U^{1,0}$ and $\tilde{z} \in U^{0,1}$, the images of $U^{1,0}$ and $U^{0,1}$ are open, and separate $[z]$ and $[\tilde{z}]$ by assumption (4). \square

The construction of Z from $\hat{Z} = P(\mathcal{L}_{1,0}^* \oplus \mathcal{L}_{0,1}^*)$ yields the diagram

$$(10) \quad \begin{array}{ccc} & \hat{Z} = P(\mathcal{L}_{1,0}^* \oplus \mathcal{L}_{0,1}^*) & \\ \phi \swarrow & \downarrow & \searrow p \\ & Z \times S^c & \\ \pi_Z \swarrow & & \searrow \pi_{S^c} \\ Z & & S^c. \end{array}$$

The induced (vertical) map $(\phi, p): \hat{Z} \rightarrow Z \times S^c$ is injective and its image is the incidence relation $F_S \subseteq F_M$ for canonical twistor lines: for $y \in S^c$, we write $u(y) := \phi(p^{-1}(y))$ for the canonical twistor line parametrized by y .

Definition 3.1. The *normal bundle* \mathcal{N} on $\hat{Z} \cong F_S$ is the bundle ϕ^*TZ/Vp , where Vp denotes the vertical bundle of $p: \hat{Z} \rightarrow S^\mathbb{C}$, with fibre $\mathcal{N}_{(z,y)} = T_z Z / T_z(u(y))$.

Proposition 3.2. $\mathcal{N} = \mathcal{N}^{1,0} \oplus \mathcal{N}^{0,1}$, where

$$\begin{aligned}\mathcal{N}^{1,0} &\cong p^*(TS^{1,0} \otimes \mathcal{L}_{1,0}^*) \otimes \mathcal{O}_{\mathcal{L}_{1,0}^* \oplus \mathcal{L}_{0,1}^*}(1), \\ \mathcal{N}^{0,1} &\cong p^*(TS^{0,1} \otimes \mathcal{L}_{0,1}^*) \otimes \mathcal{O}_{\mathcal{L}_{1,0}^* \oplus \mathcal{L}_{0,1}^*}(1).\end{aligned}$$

Proof. For any $y = (x, \tilde{x}) \in S^\mathbb{C}$, we define $(n+1)$ -dimensional submanifolds of Z by

$$\begin{aligned}\hat{Z}_{\tilde{x}}^{1,0} &= Z_{\tilde{x}}^{1,0} \cup \phi_{0,1}((\pi_{0,1} \circ p)^{-1}(\tilde{x})), \\ \hat{Z}_x^{0,1} &= Z_x^{0,1} \cup \phi_{1,0}((\pi_{1,0} \circ p)^{-1}(x)).\end{aligned}$$

By Remark 1.1, these are well defined smooth submanifolds of Z , and for any $y = (x, \tilde{x}) \in S^\mathbb{C}$, we have

$$T\hat{Z}_{\tilde{x}}^{1,0}|_{u(y)} + T\hat{Z}_x^{0,1}|_{u(y)} = TZ|_{u(y)} \quad \text{and} \quad T\hat{Z}_{\tilde{x}}^{1,0}|_{u(y)} \cap T\hat{Z}_x^{0,1}|_{u(y)} = Tu(y)$$

Hence $\mathcal{N} = \mathcal{N}^{1,0} \oplus \mathcal{N}^{0,1}$, where

$$\mathcal{N}_{(z,y)}^{1,0} = T_z \hat{Z}_{\tilde{x}}^{1,0} / T_z u(y) \quad \text{and} \quad \mathcal{N}_{(z,y)}^{0,1} = T_z \hat{Z}_x^{0,1} / T_z u(y).$$

The (canonical) identification of $\mathcal{N}^{1,0}$ with $p^*(TS^{1,0} \otimes \mathcal{L}_{1,0}^*) \otimes \mathcal{O}_{\mathcal{L}_{1,0}^* \oplus \mathcal{L}_{0,1}^*}(1)$ follows easily from Observation 2.2, as $\hat{Z}_{\tilde{x}}^{1,0}$ is a blow-down along the zero section of the projective bundle $p^{-1}(\pi_{0,1}^{-1}(\tilde{x})) \subseteq P(\mathcal{L}_{1,0}^* \oplus \mathcal{L}_{0,1}^*)$ over $\pi_{0,1}^{-1}(\tilde{x}) \xrightarrow{\pi_{1,0}} S^{1,0}$, and $\mathcal{V}_{\tilde{x}}^{1,0}/u(y) \cong T_x S^{1,0} \otimes (\mathcal{L}_{1,0}^* \otimes \mathcal{L}_{0,1})_y$. A similar argument identifies $\mathcal{N}^{0,1}$. \square

We next construct the real structure on Z . By definition the holomorphic line bundles $\overline{\mathcal{L}_{0,1}} \rightarrow \overline{S^{0,1}}$ and $\mathcal{L}_{1,0} \rightarrow S^{1,0}$ are isomorphic, and we denote the biholomorphisms $\overline{S^{0,1}} \rightarrow S^{1,0}$ and $\overline{\mathcal{L}_{0,1}} \rightarrow \mathcal{L}_{1,0}$ by θ . The real structure ρ on $S^\mathbb{C} \hookrightarrow S^{1,0} \times S^{0,1}$ sends (x, \tilde{x}) to $(\theta(\tilde{x}), \theta^{-1}(x))$. We lift this real structure to $\hat{Z} = P(\mathcal{L}_{1,0}^* \oplus \mathcal{L}_{0,1}^*)$ by defining $\rho([\sigma, \tilde{\sigma}]) = [\tilde{\sigma} \circ \theta^{-1}, -\sigma \circ \theta]$, where the minus sign ensures ρ has no fixed points. Since $\rho(\underline{0}) = \underline{\infty}$, ρ maps $\mathcal{L}_{1,0} \otimes \mathcal{L}_{0,1}^*$ to $\mathcal{L}_{1,0}^* \otimes \mathcal{L}_{0,1}$. Since the leafwise connections \mathcal{D}^∇ are (by construction) related by θ , ρ induces an antiholomorphic isomorphisms, also denoted ρ , between $\mathcal{V}^{0,1}$ and $\mathcal{V}^{1,0}$, with $\rho \circ \phi_{1,0} = \phi_{0,1} \circ \rho$ and $\rho \circ \phi_{0,1} = \phi_{1,0} \circ \rho$. We further observe (again by construction) that for any $v \in \mathcal{V}^{0,1}$,

$$\rho(\lambda \cdot v) = \rho(\lambda v) = \bar{\lambda} \rho(v) = \bar{\lambda}^{-1} \cdot \rho(v),$$

where \cdot denotes the \mathbb{C}^\times action. Thus ρ intertwines the S^1 -actions on $\mathcal{V}^{0,1}$ and $\mathcal{V}^{1,0}$.

Proposition 3.3. *We may choose $U^{0,1}$ and $U^{1,0}$ so that $Z^{0,1}$ and $Z^{1,0}$ are S^1 -invariant with $\rho(Z^{0,1}) = Z^{1,0}$. Then ρ induces an S^1 -invariant antiholomorphic involution of Z with no fixed points on any real (ρ -invariant) canonical twistor line.*

Proof. Take $U^{0,1}$ to be a sufficiently small S^1 -invariant neighbourhood of the zero section in $\mathcal{V}^{0,1}$ so that $\phi_{0,1}^{-1}(U^{0,1}) \cap \rho(\phi_{0,1}^{-1}(U^{0,1})) = \emptyset$. Now set $U^{1,0} = \rho(U^{0,1})$. The real canonical twistor lines are the images of the fibres of p over the real submanifold $S \subseteq S^\mathbb{C}$. Since $\rho \circ \phi = \phi \circ \rho$, ρ has no fixed points on any such twistor line. \square

Corollary 3.1. *Z is a twistor space, and for any canonical twistor line $u = u(y)$ (with normal bundle $\mathcal{N}|_u \cong \mathcal{N}^{1,0}|_u \oplus \mathcal{N}^{0,1}|_u$ isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$),*

$$\begin{aligned}H^0(u, \mathcal{N}^{1,0}|_u) &= (TS^{1,0} \otimes \mathcal{L}_{1,0}^*)_y \otimes (\mathcal{L}_{1,0} \oplus \mathcal{L}_{0,1})_y \\ H^0(u, \mathcal{N}^{0,1}|_u) &= (TS^{0,1} \otimes \mathcal{L}_{0,1}^*)_y \otimes (\mathcal{L}_{1,0} \oplus \mathcal{L}_{0,1})_y.\end{aligned}$$

3.2. The quaternionic manifold. By Corollary 3.1 and [6], the moduli space of twistor lines in Z is a complexified quaternionic manifold M^c with $TM^c = \mathcal{E} \otimes \mathcal{H}$, where

$$(11) \quad \begin{aligned} \mathcal{E}|_{S^c} &= (TS^{1,0} \otimes \mathcal{L}_{1,0}^*) \oplus (TS^{0,1} \otimes \mathcal{L}_{0,1}^*), & \mathcal{H}|_{S^c} &= \mathcal{L}_{1,0} \oplus \mathcal{L}_{0,1} \\ TM^c|_{S^c} &= TS^{1,0} \oplus TS^{0,1} \oplus (TS^{1,0} \otimes \mathcal{L}_{1,0}^* \otimes \mathcal{L}_{0,1}) \oplus (TS^{0,1} \otimes \mathcal{L}_{1,0} \otimes \mathcal{L}_{0,1}^*). \end{aligned}$$

Note that in this decomposition, the terms $TS^{1,0} \oplus TS^{0,1}$ correspond to the tangent space to the submanifold S^c of M^c . Furthermore, the moduli space of real twistor lines is a real quaternionic manifold M in M^c containing S [37]. Since the S^1 action on Z is generated by a holomorphic vector field, whose local flow maps twistor lines to twistor lines, it induces an S^1 action on M^c , preserving M , and fixing S^c pointwise.

Proposition 3.4. *S is a maximal totally complex submanifold of M , and the induced c -projective structure via Theorem 2 is the original c -projective structure Π_c on S .*

Proof. By [6, 37], $\mathcal{Q} \subseteq \mathfrak{gl}(TM)$ is isomorphic to the bundle of real tracefree endomorphisms of $\mathcal{H}|_M$. The real endomorphisms of $\mathcal{H}|_S = (\mathcal{L}_{1,0} \oplus \mathcal{L}_{0,1})|_S$ (see (11)) are those commuting with its quaternionic structure $(\sigma, \tilde{\sigma}) \mapsto (\tilde{\sigma} \circ \theta^{-1}, \sigma \circ \theta)$. In particular

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

is a section of \mathcal{Q} , preserving TS , and inducing the original complex structure on S . The bundle J^\perp consists of endomorphisms of \mathcal{H} of the form

$$I_s = \begin{pmatrix} 0 & -s^{-1} \\ s & 0 \end{pmatrix}.$$

where s is a unit section of $(\mathcal{L}_{1,0}^* \otimes \mathcal{L}_{0,1})|_S$. Clearly the induced endomorphisms of TM maps TS into $(TS^{1,0} \otimes \mathcal{L}_{1,0}^* \otimes \mathcal{L}_{0,1}) \oplus (TS^{0,1} \otimes \mathcal{L}_{1,0} \otimes \mathcal{L}_{0,1}^*)$. Thus (S, J) is a maximal totally complex submanifold of (M, Q) .

By Remark 2.5 the original and induced c -projective structures on S are uniquely determined by the corresponding families of holomorphic flat projective structures on the leaves of the $(1, 0)$ and $(0, 1)$ foliations of S^c . For $x \in S^{1,0}$, the original flat projective structure on $\pi_{1,0}^{-1}(x)$ has a development into $P(\mathcal{V}_x^{0,1}) \subseteq P(T_z Z)$, where z is the zero vector in $\mathcal{V}_x^{0,1}$. Hence $\pi_{1,0}^{-1}(x)$ is a projective submanifold of the α -submanifold corresponding to z (with its canonical projective structure). Hence by Proposition 2.6, any quaternionic connection on M^c induces a connection on $\pi_{1,0}^{-1}(x)$ compatible with its original projective structure. \square

Proposition 3.5. *Locally near S , M is S^1 -equivariantly diffeomorphic to a neighbourhood of the zero section of $TS \otimes \mathfrak{U}$, where $\mathfrak{U} = (\mathcal{L}_{1,0} \otimes \mathcal{L}_{0,1}^*)|_S$ is unitary.*

Proof. By (11), the normal bundle to S in M is the real part of $(TS^{1,0} \otimes \mathcal{L}_{1,0}^* \otimes \mathcal{L}_{0,1}) \oplus (TS^{0,1} \otimes \mathcal{L}_{1,0} \otimes \mathcal{L}_{0,1}^*)$. The result follows by the equivariant tubular neighbourhood theorem. \square

3.3. Proof of Theorem 5. Let (M, Q) be a quaternionic $4n$ -manifold with a quaternionic S^1 action whose fixed point set has a connected component S which is a submanifold of real dimension $2n$ with no triholomorphic points.

If J is the section of $Q|_S$ generating the infinitesimal S^1 -action, then $(TM|_S, J)$ decomposes into weight spaces for the action, with zero weight space TS . Thus TS

is J -invariant, and for any $I \in J^\perp$, ITS is a nonzero weight space, complementary to TS in TM . It follows that S is a (maximal) totally complex submanifold of M . By restricting to a neighbourhood of S in M , we may assume that the S^1 action has no other fixed points. It thus lifts to a holomorphic S^1 action on the twistor space $Z \rightarrow M$, generated by a holomorphic vector field transverse to the fibres over $M \setminus S$, tangent to the fibres over S , and vanishing (only) along the sections $\pm J$ of $Z|_S$, denoted $S^{1,0}$ and $S^{0,1}$. Let $\phi: \hat{Z} \rightarrow Z$ be the blow-up of Z along $S^{1,0} \cup S^{0,1}$, with exceptional divisor $\underline{0} \cup \underline{\infty}$, where $\underline{0}$ and $\underline{\infty}$ are the projective normal bundles in Z of $S^{1,0}$ and $S^{0,1}$ respectively. The real structure on Z (induced by $-id$ on \mathcal{Q}) interchanges $S^{1,0}$ and $S^{0,1}$, and induces a fibre-preserving real structure on \hat{Z} interchanging $\underline{0}$ and $\underline{\infty}$.

The proper transform in \hat{Z} of any fibre of $Z|_S$ is a rational curve with trivial normal bundle meeting both $\underline{0}$ and $\underline{\infty}$. Thus $\phi^{-1}(Z|_S)$ has a neighbourhood foliated by a $2n$ -dimensional moduli space S^c of rational curves with trivial normal bundle. Each such curve meets $\underline{0}$ and $\underline{\infty}$ in unique points, and projects to a twistor line in Z meeting $S^{1,0}$ and $S^{0,1}$ in unique points. The induced map $S^c \rightarrow S^{1,0} \times S^{0,1}$ is an immersion along the proper transforms of the fibres of $Z|_S$, hence an open embedding in a neighbourhood. Thus we may assume \hat{Z} is a P^1 -bundle over a complex $2n$ -manifold S^c , which embeds as an open subbundle of $\underline{0} \rightarrow S^{1,0}$ and $\underline{\infty} \rightarrow S^{0,1}$, and as an open neighbourhood S^c of the diagonal in $S^{1,0} \times S^{0,1}$. By Lemma 1.1 and Proposition 2.6, the induced c-projective structure on S has c-projective curvature of type $(1, 1)$: in the complexified c-projective structure on S^c , the fibres over $S^{1,0}$ and $S^{0,1}$ are projectively-flat.

The holomorphic S^1 action on Z has a single nontrivial weight space at each point of $S^{1,0} \cup S^{0,1}$ (the normal bundle to S in M has the same weight as the normal bundle to $S^{1,0}$ or $S^{0,1}$ in $Z|_S$). Hence it acts by scalar multiplication on the normal bundles $\mathcal{V}^{0,1}$ to $S^{1,0}$ in Z , and $\mathcal{V}^{1,0}$ to $S^{0,1}$ in Z . In particular, the S^1 action is trivial on the projectivizations of $\mathcal{V}^{0,1}$ and $\mathcal{V}^{1,0}$, i.e., the lifted action on \hat{Z} fixes $\underline{0} \cup \underline{\infty}$ pointwise. Thus $\hat{Z} \setminus (\underline{0} \cup \underline{\infty})$ is a holomorphic principal \mathbb{C}^\times -bundle over S^c , with associated P^1 -bundle \hat{Z} . The associate (dual) line bundles are subbundles of the pullbacks of $\mathcal{V}^{0,1}$ and $\mathcal{V}^{1,0}$ to S^c , which thus have trivial Cartan connections along the fibres over $S^{1,0}$ and $S^{0,1}$ respectively. Unravelling the constructions in §2.6, these are twists of the Cartan connections induced by the c-projective structure by dual and conjugate line bundles which are flat along the fibres over $S^{1,0}$ and $S^{0,1}$; we deduce that these twists come from a complex line bundle $\mathcal{L} \rightarrow S$ with both a holomorphic and an antiholomorphic structure, hence a (Chern) connection with curvature of type $(1, 1)$. We now have reconstructed the data for the quaternionic Feix–Kaledin construction of Z as a blow-down on \hat{Z} , and hence of (a neighbourhood of S in) M .

4. EXAMPLES AND APPLICATIONS

4.1. The hypercomplex and hyperkähler cases. The line bundles $\mathcal{L}_{1,0} \rightarrow S^{1,0}$ and $\mathcal{L}_{0,1} \rightarrow S^{0,1}$ which provide the input to the quaternionic Feix–Kaledin construction are twists of the line bundle $\mathcal{O}_S(1)$, over a c-projective manifold S with c-projective curvature of type $(1, 1)$, by a connection ∇ on a complex line bundle $\mathcal{L} \rightarrow S$ with curvature of type $(1, 1)$. When $\mathcal{O}_S(1)$ itself admits such a connection, we can take $\mathcal{L} = \mathcal{O}_S(1)^*$, so that $\mathcal{L}_{1,0} \rightarrow S^{1,0}$ and $\mathcal{L}_{0,1} \rightarrow S^{0,1}$ are trivial bundles.

Proposition 4.1. *If the c-projective structure Π_c on S admits a real-analytic connection D with curvature of type $(1, 1)$, and ∇ is the induced connection on $\mathcal{L} = \mathcal{O}_S(-1)$, then*

the quaternionic manifold M of Theorem 4 is hypercomplex, and is the hypercomplex manifold constructed by Feix [19]. Furthermore, when D is the Levi-Civita connection of a Kähler metric, then M is hyperkähler, as in [18].

Proof. As noted above, the assumptions of this theorem imply that $\mathcal{L}_{1,0} \rightarrow S^{1,0}$ and $\mathcal{L}_{0,1} \rightarrow S^{0,1}$ are trivial. We compute their spaces of affine sections using the connection $D \in \Pi_c$, so that twisted connections D^∇ on $\mathcal{L}_{1,0}$ and $\mathcal{L}_{0,1}$ are trivial. Furthermore, D has curvature of type $(1,1)$ if and only if Π_c has c-projective curvature of type $(1,1)$ and r^D has type $(1,1)$. Thus, in this case, r^D vanishes on the leaves of the $(1,0)$ and $(0,1)$ foliations, and hence a function f on such a leaf defines an affine section if and only if $Ddf = 0$ along the leaf, i.e., f is an affine function with respect to the flat affine connection induced by D on the leaf. We conclude that $\mathcal{V}^{1,0}$ and $\mathcal{V}^{0,1}$ are vector bundles dual to the spaces of affine functions along leaves considered by Feix [18, 19].

It is easy to check that $\phi_{1,0}: S^c \times \mathbb{C} \rightarrow \mathcal{V}^{1,0}$ and $\phi_{0,1}: S^c \times \mathbb{C} \rightarrow \mathcal{V}^{0,1}$ send $(x, \tilde{x}, 1)$ to the evaluation maps that Feix uses in her construction; hence our construction reduces to hers. Because constant functions are affine, the projection $\hat{Z} = S^c \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ descends to Z , which implies M is hypercomplex by [23]. We refer to [18] for the proof that M is hyperkähler when D is the Levi-Civita connection of a Kähler metric. \square

4.2. The four-dimensional case and Einstein–Weyl spaces. In four dimensions, a quaternionic manifold (M, \mathcal{Q}) is a self-dual conformal manifold. LeBrun [34] studied quotients of self-dual manifolds by a class of S^1 actions which he called “docile”; these include *semi-free* S^1 actions (whose stabilizers are either trivial or the whole group), for which one of his results specializes as follows.

Lemma 4.1 ([34]). *Let (M, g) be a self-dual manifold with a semi-free S^1 action with a nonempty surface S of fixed points, and no isolated fixed points. Let B be a maximal smooth manifold (without boundary) in $Y = M/S^1$. Then the Einstein–Weyl structure [21] on B defined by the Jones–Tod correspondence [26] has S as an asymptotically hyperbolic end.*

This means that the Weyl structure is asymptotic (in a precise sense [34]) to the Levi-Civita connection of the hyperbolic metric in a punctured neighbourhood of the image of S in Y .

Proposition 4.2. *The quotient by the S^1 action of the self-dual conformal 4-manifold obtained by the quaternionic Feix–Kaledin construction is Einstein–Weyl with S as an asymptotically hyperbolic end.*

Proof. The S^1 action is induced by a holomorphic vector field on the twistor space, which implies that it is conformal (see for example [26]). It is also clearly semi-free and the zero section is the fixed point set, which by completes the proof. \square

There are special features of the quaternionic Feix–Kaledin construction of (M, \mathcal{Q}) from a surface S with a c-projective structure. As discussed in §2.8, such a surface S carries more data than (J, Π_c) . In the approach discussed there, the additional data is a second order operator [13]. Alternatively, one can characterize the Cartan connection on S or S^c explicitly. Following [9, 11], we now consider the latter approach (on S^c).

A *conformal Cartan connection* $(\mathcal{V}, A, \mathcal{D})$ on a holomorphic surface S^c consists of:

- a rank 4 holomorphic vector bundle $\mathcal{V} \rightarrow S^c$ with inner product \langle, \rangle ;
- a null line subbundle $A \subset \mathcal{V}$;

- a linear metric connection \mathcal{D} satisfying the Cartan condition, that $\mathcal{D}|_\Lambda \bmod \Lambda$ is an isomorphism from $TS^c \otimes \Lambda$ to Λ^\perp/Λ .

The Cartan condition implies that TS^c carries a conformal structure. We may suppose that $S^c \hookrightarrow S^{1,0} \times S^{0,1}$ where the leaves of the $(1,0)$ and $(0,1)$ foliations are the null curves of the conformal structure; we then write $\Lambda^\perp = U^+ + U^-$, where $U^+ \cap U^- = \Lambda$ and $\mathcal{D}^{1,0}\Lambda \subseteq U^+$ and $\mathcal{D}^{0,1}\Lambda \subseteq U^-$. Observe that $\mathcal{D}^{1,0}$ and $\mathcal{D}^{0,1}$ are flat connections, on U^+ and U^- respectively, along the curves of the $(1,0)$ and $(0,1)$ foliations respectively.

In [9], the first author constructed a minitwistor space [21] of an asymptotically hyperbolic Einstein–Weyl manifold B from a conformal Cartan connection by lifting the curves of $(1,0)$ and $(0,1)$ foliations to $P(U^+)$ and $P(U^-)$ respectively, and gluing together the leaf spaces. We now relate this approach to the quaternionic Feix–Kaledin construction. The work of [9] already shows that B is a quotient of a selfdual 4-manifold M with an S^1 action, whose twistor space Z is also constructed explicitly there. Hence it suffices to establish the following.

Proposition 4.3. *The construction of the twistor space in [9] from S coincides with the quaternionic Feix–Kaledin construction given here.*

Proof. The inner product on \mathcal{V} induces a duality between U^+ and \mathcal{V}/U^+ , with respect to which $\mathcal{D}^{1,0}$ induces dual connections along the curves of the $(1,0)$ foliation. We thus have isomorphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & T^*S^{1,0} \otimes \mathcal{V}/\Lambda^\perp & \longrightarrow & J^1(\mathcal{V}/\Lambda^\perp) & \longrightarrow & \mathcal{V}/\Lambda^\perp \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \Lambda^\perp/U^+ & \longrightarrow & \mathcal{V}/U^+ & \longrightarrow & \mathcal{V}/\Lambda^\perp \longrightarrow 0, \end{array}$$

and similarly for $\mathcal{D}^{0,1}$ on U^- and \mathcal{V}/U^- along the $(0,1)$ foliation.

As explained in [9], we may also suppose that $\Lambda = \Lambda^+ \otimes \Lambda^-$, with Λ^+ and Λ^- trivial along the $(1,0)$ and $(0,1)$ foliations respectively. The bundles $\tilde{U}^+ := U^+ \otimes (\Lambda^+)^{-2}$ and $\tilde{U}^- := U^- \otimes (\Lambda^-)^{-2}$ have induced flat connections along the $(1,0)$ and $(0,1)$ foliations respectively, dual to $(\mathcal{V}/U^-) \otimes (\Lambda^+)^2$ and $(\mathcal{V}/U^+) \otimes (\Lambda^-)^2$. Hence, along the null curves, the spaces \mathcal{V}^\pm of parallel sections of \tilde{U}^\pm are dual to spaces of affine sections of $(\mathcal{V}/\Lambda^\perp) \otimes (\Lambda^\pm)^2 \cong \Lambda_\pm \otimes (\Lambda_\mp)^*$. Hence the construction in [9] reduces to the one herein by taking $\Lambda^+ = \mathcal{L}_{0,1}^*$ and $\Lambda^- = \mathcal{L}_{1,0}^*$. \square

The link with conformal Cartan connections elucidates the role of the connection ∇ on $\mathcal{L} \rightarrow S$: any conformal Cartan connection over S , is up to isomorphism, the twist of the normal Cartan connection (induced by a Möbius structure) [13] by such a connection ∇ . The construction of the Einstein–Weyl manifold B as an S^1 -quotient equips it with a distinguished gauge (or abelian monopole) [26]. Since $P(\mathcal{E} \otimes \mathcal{L}) = P(\mathcal{E})$ for any line bundle \mathcal{L} and a vector bundle \mathcal{E} , the construction of the minitwistor space (from $P(\mathcal{V}^+)$ and $P(\mathcal{V}^-)$) does not depend on the choice of (\mathcal{L}, ∇) . We thus have a gauge for each such choice.

4.3. Complex grassmannians. In [43], J. Wolf classified the totally complex submanifolds of quaternionic symmetric spaces fixed by a circle action. These provide many examples of the quaternionic Feix–Kaledin construction which are not (even locally) hypercomplex. We focus on the quaternionic symmetric spaces isomorphic (for some $n \geq 1$) to $Gr_2(\mathbb{C}^{n+2})$, the complex grassmannian of 2-dimensional subspaces of \mathbb{C}^{n+2} . The twistor space Z is the flag manifold $F_{1,n+1}(\mathbb{C}^{n+2})$ of pairs $B \subseteq W \subseteq \mathbb{C}^{n+2}$

with $\dim B = 1$ and $\dim W = n + 1$. The standard hermitian inner product $\langle \cdot, \cdot \rangle$ on \mathbb{C}^{n+2} defines a real structure on Z , sending the flag $B \subseteq W$ to $W^\perp \subseteq B^\perp$. It also defines an antiholomorphic diffeomorphism between $Gr_2(\mathbb{C}^n)$ with $Gr_n(\mathbb{C}^{n+2})$, and it is convenient to identify the quaternionic manifold M with the graph of this map in $Gr_2(\mathbb{C}^n) \times Gr_n(\mathbb{C}^{n+2})$. In these terms the twistor projection from Z to M , whose fibres are the real twistor lines, sends $B \subseteq W$ to the pair $(B \oplus W^\perp, B^\perp \cap W)$ in M .

The space of all twistor lines in Z is the holomorphic (i.e., complexified) quaternionic manifold $M^c \cong \{(U, V) \in Gr_2(\mathbb{C}^{n+2}) \times Gr_n(\mathbb{C}^{n+2}) : \mathbb{C}^{n+2} = U \oplus V\}$:

- the flags $B \subseteq W$ on the twistor line corresponding to $(U, V) \in M^c$ have $B \subseteq U$ and $V \subseteq W$, so that $B = U \cap W$ and $W = V + B$;
- this twistor line is canonically isomorphic to $P(U) \cong P(\mathbb{C}^{n+2}/V)$;
- also $\mathcal{O}_U(-1) \cong \mathcal{O}_{\mathbb{C}^{n+2}/V}(-1)$ via the map sending $b \in U$ to $b + V$ in \mathbb{C}^{n+2}/V .

A fixed decomposition $\mathbb{C}^{n+2} = A \oplus \tilde{A}$, with $\dim A = 1$ and $\dim \tilde{A} = n + 1$, determines a submanifold $S^c = \{(U, V) \in M^c : A \subseteq U, V \subseteq \tilde{A}\}$ of M^c :

- $(U, V) \mapsto (U/A, V)$ embeds S^c as an open subset of $P(\mathbb{C}^{n+2}/A) \times Gr_n(\tilde{A})$;
- the fibre of S^c over $V \subseteq \tilde{A}$ is isomorphic to the affine space $P(\mathbb{C}^{n+2}/A) \setminus P((V \oplus A)/A)$ and similarly for the fibre over $U \supseteq A$;
- $P(\mathbb{C}^{n+2}/A) \cong P(\tilde{A})$ may be identified with $S^{1,0} = \{B \subseteq \tilde{A} : \dim B = 1\} \subseteq Z$, and, similarly, $Gr_n(\tilde{A}) \cong Gr_n(\mathbb{C}^{n+2}/A)$ with $S^{0,1} = \{A \subseteq W : \dim W = n + 1\} \subseteq Z$.
- $Gr_n(\tilde{A}) \cong P(\tilde{A}^*)$ is the dual projective space to $P(\mathbb{C}^{n+2}/A) \cong P(\tilde{A})$, and for any $(U, V) \in S^c$, the corresponding tautological lines $(\tilde{A}/V)^* \cong V^0 \subseteq \tilde{A}^*$ and $U/A \cong U \cap \tilde{A}$ are canonically dual to each other.

If $\tilde{A} = A^\perp$ then the real points in $S^c \subseteq M^c$ form a maximal totally complex submanifold $S \subseteq M$ fixed by an S^1 action, and $S^{1,0}, S^{0,1}$ are lifts of S to Z with respect to the induced complex structures $\pm J$ on S . Hence Theorem 5 applies.

Following the proof in §3.3, let \hat{Z} be the blow-up of Z along $S^{1,0} \cup S^{0,1}$. The fibre of $\hat{Z} \rightarrow S^c$ over (U, V) is $P(U) \cong P(\mathbb{C}^{n+2}/V)$, and the natural map to Z is a biholomorphism over $(B \subseteq W) \in Z$ unless $B = A$ or $W = \tilde{A}$, which are the “zero” and “infinity” sections $\underline{0}$ and $\underline{\infty}$ of $\hat{Z} \rightarrow S^c$, mapping to $S^{1,0}$ and $S^{0,1}$ respectively. Identifying $S^c = P(\tilde{A}) \times P(\tilde{A}^*)$, $\hat{Z} \cong P(\mathcal{O}_{\tilde{A}}(-1) \oplus \mathcal{O})|_{S^c} \cong P(\mathcal{O} \oplus \mathcal{O}_{\tilde{A}^*}(-1))|_{S^c}$.

We now set $\tilde{A} = A^\perp$ and identify $P(\tilde{A}^*)$ with $\overline{P(A^\perp)}$ using the real structure; thus S^c is the open subset $\{([\ell], [w]) \in P(A^\perp) \times \overline{P(A^\perp)} : \langle \ell, w \rangle \neq 0\}$, and the hermitian metric induces a pairing of the tautological line bundles over $P(A^\perp)$ and $\overline{P(A^\perp)}$, i.e., a nonvanishing section of $\mathcal{O}(1, 1) \rightarrow S^c$. On the (anti-)diagonal S in $P(A^\perp) \times \overline{P(A^\perp)}$, this section may be viewed as a hermitian metric on $\mathcal{O}(-1) \rightarrow S$.

Locally, $\mathcal{O}(-1) \rightarrow S$ has a square root $\mathcal{L} = \mathcal{O}(-\frac{1}{2})$, and the trivialization of $\mathcal{O}(1, 1)$ identifies $\mathcal{O}(1, 0)$ with $\mathcal{O}(\frac{1}{2}, -\frac{1}{2})$. Thus we have the following result.

Proposition 4.4. *Let Π_c be the flat c -projective structure on S and let $\mathcal{L} = \mathcal{O}(-\frac{1}{2})$ (defined over any open subset of S). The standard hermitian metric on \mathbb{C}^{n+2} induces hermitian metric on \mathcal{L} with Chern connection ∇ . Then Z and M are obtained from the quaternionic Feix–Kaledin construction applied to these data.*

4.4. Further directions. The construction presented in this paper prompts several directions for further study.

- In [3], Alekseevsky and Marchiafava study in particular the geometry of maximal totally complex submanifolds of quaternionic Kähler manifolds. In view of Lemma 1.1,

it would be natural to study such submanifolds S of (general) quaternionic manifolds M in the context of parabolic submanifold geometry [10]. In particular, for the submanifolds appearing in the quaternionic Feix–Kaledin construction, we would like to read off properties of the c-projective structure Π_c and connection ∇ from the extrinsic geometry of S in M .

- The quaternionic Feix–Kaledin construction produces a hyperkähler metric or hypercomplex structure when it reduces to the original constructions by Feix and Kaledin. It is natural to ask, more generally, when the quaternionic manifold M admits hypercomplex structures or quaternionic Kähler metrics. In [8], the first author studied the relationship between compatible complex structures [4] on M , quaternionic connections, and the geometry of the c-projective structure Π_c .
- The work of Haydys and Hitchin [20, 22] shows that quaternionic Kähler metrics with S^1 actions come in (one parameter) families. The construction presented here has a similar property: for a given c-projective manifold (S, Π_c) , one can vary the line bundle with connection (\mathcal{L}, ∇) , for instance by taking tensor powers. The work of Feix [19] already suggests that the choice of line bundle \mathcal{L} on S is connected with the hyperholomorphic bundle appearing in the Haydys–Hitchin correspondence. As a special case, this correspondence relates semi-flat hyperkähler metrics (i.e., the rigid c-map) to the c-map construction of quaternionic Kähler metrics [1, 17, 22, 35]. It would be natural to extend these ideas to more general quaternionic manifolds, using the Swann bundle and the work of Joyce [27, 41].

We hope to return to some of these directions in subsequent work.

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